Vibrational Analysis of Foundation Structures with Elliptic Cross Sections in Elastic Ground

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Abstract

This paper deals with the three dimensional analysis of foundation structures with elliptic cross sections when the surface layer is excited by a sinusoidal disturbing force from the bedrock. First, the effect of earth pressure on an elliptic cylinder is investigated, and then the analysis of the frequency response for the displacement of the elliptic cylinder is discussed.

Our analysis using the elliptic cylindrical coordinates showed that the investigations in the circular cylindrical coordinates are a special case of our analysis and that not only the difference in the direction of vibration but also the shape of the foundation structures has a great effect upon the frequency response of foundation structures in a semi-infinite elastic stratum.

1. Introduction

So far various theoretical and experimental studies have been made with respect to the aseismic design of bridge piers. At present, however, a reasonable method of aseiemic design has not yet been established. This is mainly due to the fact that the physical properties of the surface layer around the foundation structures are still indistinct. For this reason, the surface layer has often been treated, for convenience' sake, as an idealized mechanical system represented by the linear or non-linear spring constant or the modulus of foundation. These coefficients, however, have no relation with the shapes of the cross section of the structures, and they neglect the effects of dynamic earth pressure acting on the foundation structures. It is clearly the complicated behaviour of the ground around the structures during an earthquakes that makes the seismic response analysis of underground structures difficult. Accordingly it would not be too much to say that the seismic response of a foundation structure is best analyzed if we can correctly estimate the influence of the dynamic earth pressure on the underground structure.

From these points of view, Prof. Tajimi developed an analysis of the frequency response of foundations by treating the surface layer as a homogenious isotropic elastic medium. From his study, however, we can not learn the effect of the difference in the cross section of the foundation structures because the object of his analysis is confined to circular cylinders and the effect of the length of foundation structures on the frequency response.

In this paper we examined what influence the variation of the cross section from circle to thin plate has on the frequency response of foundation structures.

2. Response Analysis of Surface Layer and Elliptic Cylinder

(1) Equation of Motion for the Surface Layer

Assumptions about the surface layer are the same as those made in the analysis by Prof. Tajimi as follows:-(1) The surface layer is a homogeneous isotropic elastic medium and is supported by the bedrock. (2) Viscous damping is not

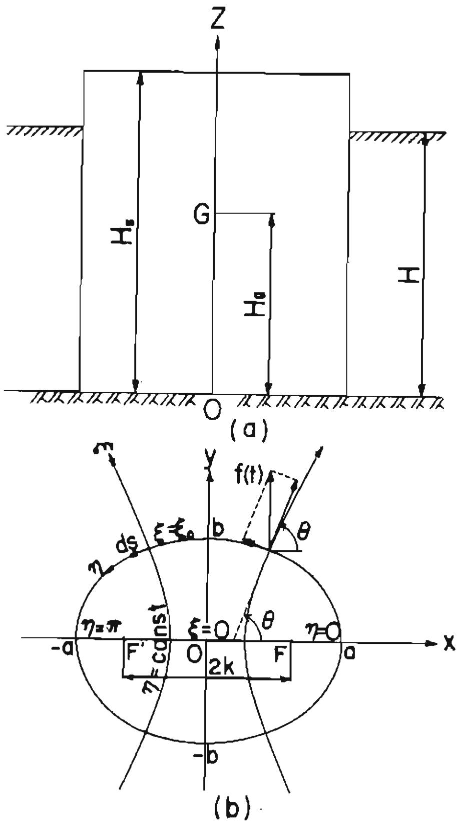

Fig. 1 Model of Rigid Foundation Structure and Cross Section

taken into account. (3) Vertical displacement is neglected because it is small in comparison with the displacement of the horizontal component.

An elliptic cylinder in a surface layer of thickness H, the bottom of which is supported by the bedrock, is shown in Fig. 1 (a). Its elliptic cross section with major axis $2a$ and minor axis $2b$ is shown in Fig. 1 (b). Transformation from the cartesian coordinates (x, y, z) into the elliptic cylindrical coordinates (ξ, η, z) leads to

$$\left.\begin{aligned} x&=k\cosh\xi\ \cos\eta \\ y&=k\sinh\xi\ \sin\eta \\ z&=z \end{aligned}\right\} \qquad \cdots\cdots\cdots(1)$$

where $k=\sqrt{a^2-b^2}$, and $2k$ is the length of the interfocal line of ellipse.

By use of eq. (1), the coordinate ξ_0 of the surface of the elliptic cylinder is obtained as eq. (2).

$$\xi_0=\tanh^{-1}\frac{b}{a}=\frac{1}{2}\ln\frac{a+b}{a-b} \qquad \cdots\cdots\cdots(2)$$

If $\xi_0=0$, the coordinate ξ_0 tends to the interfocal line $2k$. When the input ground motion $u_g=u_0e^{i\omega t}$ imparted from the bedrock excites the system in the direction of the minor axis, or the direction of the y axis of ellipse, the equations of motion of the elastic surface layer are written as follows:

$$(\lambda+2\mu)\frac{\partial\varDelta}{l\partial\xi}-2\mu\frac{\partial\tilde{\omega}_z}{l\partial\eta}+2\mu\frac{\partial(l\tilde{\omega}_\eta)}{l\partial z}=\rho\left(\frac{\partial^2u_\xi}{\partial t^2}-u_0\omega^2e^{i\omega t}\sin\theta\right) \qquad \cdots\cdots(3)$$

$$(\lambda+2\mu)\frac{\partial\varDelta}{l\partial\eta}-2\mu\frac{\partial(l\tilde{\omega}_\xi)}{l\partial z}+2\mu\frac{\partial\tilde{\omega}_z}{l\partial\xi}=\rho\left(\frac{\partial^2u_\eta}{\partial t^2}-u_0\omega^2e^{i\omega t}\cos\theta\right) \qquad \cdots\cdots(4)$$

where l, dilatation $\varDelta$ and rotations $\tilde{\omega}_\xi$, $\tilde{\omega}_\eta$, $\tilde{\omega}_z$ are represented by

$$\left.\begin{aligned} &l=k\sqrt{\cosh^2\xi-\cos^2\eta} \\ &\varDelta=\frac{\partial(lu_\xi)}{l^2\partial\xi}+\frac{\partial(lu_\eta)}{l^2\partial\eta},\quad 2\tilde{\omega}_\xi=-\frac{\partial u_\eta}{\partial z}, \\ &2\tilde{\omega}_\eta=\frac{\partial u_\xi}{\partial z},\quad 2\tilde{\omega}_z=\frac{\partial(lu_\eta)}{l^2\partial\xi}-\frac{\partial(lu_\xi)}{l^2\partial\eta} \end{aligned}\right\} \qquad \cdots\cdots\cdots(5)$$

Besides, the relations between θ and coordinates ξ, η are

$$\left.\begin{aligned} \cos\theta&=\frac{\partial x}{l\partial\xi}=\frac{\partial y}{l\partial\eta}=\frac{k\sinh\xi\cos\eta}{l} \\ &=\frac{b\cos\eta}{l} \quad (\text{at } \xi=\xi_0) \end{aligned}\right\} \qquad \cdots\cdots\cdots(6)$$

$$\left.\begin{aligned} \sin\theta&=\frac{\partial y}{l\partial\xi}=-\frac{\partial x}{l\partial\eta}=\frac{k\cosh\xi\sin\eta}{l} \\ &=\frac{a\sin\eta}{l}. \quad (\text{at } \xi=\xi_0) \end{aligned}\right\} \qquad \cdots\cdots\cdots(7)$$

Before solving eqs. (6), (7) the boundary conditions for the surface layer are set up as follows:

$$\left.\begin{array}{l}\text{(i) } \eta=0 \,:\, u_\xi=0,\ \text{(ii) } \eta=\pi/2 \,:\, u_\eta=0,\\ \text{(iii) continuity of horizontal displacements of the surface layer and}\\ \text{the elliptic cylinder at } \xi=\xi_0,\ \text{(iv) } \xi=\infty:\ \sigma_\xi=\sigma_\eta=\tau_{\xi\eta}=0,\\ \text{(v) } z=0 \,:\, u_z=0,\ \text{(vi) } z=H \,:\, \tau_{\xi z}=\tau_{\eta z}=0.\end{array}\right\} \quad \cdots\cdots(8)$$

On the other hand, the boundary conditions for the surface layer which vibrates in the direction of the major axis are given as well as in the direction of the minor axis as follows:

$$\left.\begin{array}{l}\text{(i) } \eta=0 \,:\, u_\eta=0,\ \text{(ii) } \eta=\pi/2 \,:\, u_\xi=0,\\ \text{(iii) continuity of horizontal displacements of the surface layer and}\\ \text{the elliptic cylinder at } \xi=\xi_0,\ \text{(iv) } \xi=\infty:\ \sigma_\xi=\sigma_{\xi\eta}=\tau_{\xi\eta}=0,\\ \text{(v) } z=0 : u_z=0,\ \text{(vi) } z=H : \tau_{\xi z}=\tau_{\eta z}=0.\end{array}\right\} \quad \cdots\cdots(9)$$

The boundary conditions (v) of eqs. (8), (9) are always satisfied because of the assumptions stated previously. Using the boundary conditions (8), the equations of motion for the surface layer, eqs. (3) and (4), can be solved in the following way.

Applying the divergence operation to eqs. (3), (4), we obtain

$$(\lambda+2\mu)\nabla^2\Delta+\mu\frac{\partial^2\Delta}{\partial z^2}=\rho\frac{\partial^2\Delta}{\partial t^2} \quad \cdots\cdots(10)$$

Similarly, applying the curl operation to eqs. (3), (4) we obtain

$$\mu\nabla^2(2\tilde{\omega}_z)+\mu\frac{\partial^2(2\tilde{\omega}_z)}{\partial z^2}=\rho\frac{\partial^2(2\tilde{\omega}_z)}{\partial t^2} \quad \cdots\cdots(11)$$

in which

$$\nabla^2=\frac{\partial^2}{l^2\partial\xi^2}+\frac{\partial^2}{l^2\partial\eta^2}$$

The solutions of eqs. (10) and (11) are clearly separable with respect to the variables ξ, η, z and t. Making use of the potential functions Φ and Ψ, the solutions for the displacements satisfying the boundary condition (iv) of eq. (8) may be obtained as follows:

$$u_\xi=\frac{1}{l}\left(\frac{\partial\Phi}{\partial\xi}-\frac{\partial\Psi}{\partial\eta}\right)\sin\frac{m\pi z}{2H}e^{i\omega t},\quad (m=1,3,5,\cdots\cdots) \quad \cdots\cdots(12)$$

$$u_\eta=\frac{1}{l}\left(\frac{\partial\Phi}{\partial\eta}+\frac{\partial\Psi}{\partial\xi}\right)\sin\frac{m\pi z}{2H}e^{i\omega t},\quad (m=1,3,5,\cdots\cdots) \quad \cdots\cdots(13)$$

Substituting eqs. (12), (13) into eqs. (10), (11) gives, by virtue of eq. (5),

$$\Delta=\nabla^2\Phi\ \sin\frac{m\pi z}{2H}e^{i\omega t},\quad (m=1,3,5,\cdots\cdots) \quad \cdots\cdots(14)$$

$$2\tilde{\omega}_z=\nabla^2\Psi\ \sin\frac{m\pi z}{2H}e^{i\omega t},\quad (m=1,3,5,\cdots\cdots) \quad \cdots\cdots(15)$$

Therefore eqs. (10), (11) become

$$\nabla^2\Phi+\alpha^2{}_m\Phi=0 \quad \cdots\cdots(16)$$

$$\nabla^2\Psi+\beta^2{}_m\Psi=0 \quad \cdots\cdots(17)$$

in which

$$\alpha_m=\frac{\pi}{2H}\frac{v_t}{v_l}\xi_m,\quad \beta_m=\frac{\pi}{2H}\xi_m,\quad \xi_m=\sqrt{\left(\frac{\omega}{\omega_g}\right)^2-m^2},\quad (m=1,3,5,\cdots\cdots) \quad \cdots\cdots(18)$$

Putting $\Phi=R(\xi)\cdot\Theta(\eta)$ with the aid of the separation constant λ the partial differential equation (16) may be separated into the following two ordinary differential equations:

$$\frac{d^2R}{d\xi^2}+\left(-\lambda+\frac{\alpha_m{}^2k^2}{2}\cosh 2\xi\right)R=0 \qquad \cdots\cdots(19)$$

$$\frac{d^2\Theta}{d\eta^2}+\left(\lambda-\frac{\alpha_m{}^2k^2}{2}\cos 2\eta\right)\Theta=0 \qquad \cdots\cdots(20)$$

The solutions of eqs. (19), (20) are the modified Mathieu function, and the Mathieu function, respectively[3]. The modified Mathieu functions can be represented in terms of functions of complex variables analogous to the Hankel functions in circular cylindrical coordinates; i.e.,

$$Me_{2n+1}^{(1),(2)}(\xi,q)=(p_{2n+1}/A_1^{(2n+1)})\sum_{r=1}^{\infty}(-1)^r A_{2r+1}^{(2n+1)}[J_r(v_1)H_{r+1}^{(1),(2)}(v_2)+J_{r+1}(v_1)H_r^{(1),(2)}(v_2)],$$

for $(\lambda=a_{2n+1})$

$$Ne_{2n+1}^{(1),(2)}(\xi,q)=(s_{2n+1}/B_1{}^{(2n+1)})\sum_{r=1}^{\infty}(-1)^r B_{2r+1}^{(2n+1)}[J_r(v_1)H_{r+1}^{(1),(2)}(v_2)-J_{r+1}(v_1)H_r^{(1),(2)}(v_2)],$$

for $(\lambda=b_{2n+1})$

$$Me_{2n}^{(1),(2)}(\xi,q)=(p_{2n}/A_0{}^{(2n)})\sum_{r=0}^{\infty}(-1)^r A_{2r}^{(2n)}J_r(v_1)H_r^{(1),(2)}(v_2), \qquad \text{for } (\lambda=a_{2n})$$

$$Ne_{2n+2}^{(1),(2)}(\xi,q)=-(s_{2n+2}/B_2{}^{(2n+2)})\sum_{r=0}^{\infty}(-1)^r B_{2r+2}^{(2n+2)}[J_r(v_1)H_{r+2}^{(1),(2)}(v_2)-J_{r+2}(v_1)H_r^{(1),(2)}(v_2)],$$

for $(\lambda=b_{2n+2})$

in which the coefficients $A_{2r}^{(2n+2)}$, $A_{2r+1}^{(2n+1)}$, $B_{2r+2}^{(2n+2)}$ and $B_{2r+1}^{(2n+1)}$ contain parameters q and m. The separation constant λ is shown as a_n or b_n, which is a so-called characteristic number.

On the other hand, the Mathieu functions are given by

$$ce_{2n}(\eta,q)=\sum_{r=0}^{\infty}A_{2r}^{(2n)}\cos 2r\eta, \qquad (\lambda=a_{2n})$$

$$ce_{2n+1}(\eta,q)=\sum_{r=0}^{\infty}A_{2r+1}^{(2n+1)}\cos(2r+1)\eta, \qquad (\lambda=a_{2n+1})$$

$$se_{2n+1}(\eta,q)=\sum_{r=0}^{\infty}B_{2r+1}^{(2n+1)}\sin(2r+1)\eta, \qquad (\lambda=b_{2n+1})$$

$$se_{2n+2}(\eta,q)=\sum_{r=0}^{\infty}B_{2r+2}^{(2n+2)}\sin(2r+2)\eta, \qquad (\lambda=b_{2n+2})$$

Since the surface layer is a semi-infinite elastic stratum in this study, the solution of eq. (19) should be the modified Mathieu functions of the second kind $Me_{2n+1}^{(2)}(\xi,q)$ or $Ne_{2n+1}^{(2)}(\xi,q)$ which correspond to diverging waves. Thus for the case of vibration in the direction of the minor axis we get the solutions of eqs. (16), (17) which satisfy the boundary conditions (i), (ii) and (vi) of eq. (8):

$$\Phi=C_m Ne_{2n+1}^{(2)}(\xi,q_1)se_{2n+1}(\eta,q_1) \qquad \cdots\cdots(21)$$

$$\Psi=D_m Me_{2n+1}^{(2)}(\xi,q_2)ce_{2n+1}(\eta,q_2) \qquad \cdots\cdots(22)$$

in which C_m, D_m are undetermined coefficients. The coefficients $A_{2r+1}^{(2n+1)}$ and $B_{2r+1}^{(2n+1)}$ consist of q_2, m and q_1, m respectively, where q_1, q_2 are shown as follows:

$$\left.\begin{aligned} q_1 &= \frac{\alpha^2{}_m k^2}{4} = \frac{k^2}{4}\left(\frac{\pi}{2H}\right)^2\left(\frac{v_t}{v_l}\right)^2\xi^2{}_m \\ q_2 &= \frac{\beta^2{}_m k^2}{4} = \frac{k^2}{4}\left(\frac{\pi}{2H}\right)^2\xi^2{}_m \end{aligned}\right\} \qquad \cdots\cdots\cdots(23)$$

Then, the modified Mathieu functions $\overset{(2)}{Me}_{2n+1}(\xi, q)$, $\overset{(2)}{Ne}_{2n+1}(\xi, q)$, are to be replaced by the following monotonous decreasing functions if arguments q_1, q_2 are negative:

$$Fek_{2n+1}(\xi, -q') = (s_{2n+1}/\pi B_1{}^{(2n+1)})\sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)}[I_r(v_1)K_{r+1}(v_2) - I_{r+1}(v_1)K_r(v_2)], \quad (\lambda = a_{2n+1})$$

$$Gek_{2n+1}(\xi, -q') = (p'_{2n+1}/\pi A_1{}^{(2n+1)})\sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)}[I_r(v_1)K_{r+1}(v_2) + I_{r+1}(v_1)K_r(v_2)], \quad (\lambda = b_{2n+1})$$

in which $q' = -q$ if $q < 0$.

Since the solutions of eqs. (14), (15) giving dilatation Δ and rotation $2\tilde{\omega}_z$ are not complete solutions of eqs. (3) and (4), we get complete solutions by adding to them the solution for the horizontal transverse vibration which satisfies the conditions $\Delta = 2\tilde{\omega}_z = 0$; i. e., the displacements u_ξ and u_η expanded in the Fourier sine series are represented by

$$\left.\begin{aligned} u_\xi = &\sum_{m=1,3,5,\cdots}^{s} \frac{1}{l}\Big\{C_m \dot{\overset{(2)}{Ne}}_{2n+1}(\xi, q_1)se_{2n+1}(\eta, q_1) - D_m \overset{(2)}{Me}_{2n+1}(\xi, q_2)c\dot{e}_{2n+1}(\eta, q_2) \\ &\qquad - \frac{4u_0 k\cosh\xi}{m\pi\xi^2{}_m}\left(\frac{\omega}{\omega_g}\right)^2\sin\eta\Big\}\sin\frac{m\pi z}{2H}\cdot e^{i\omega t} \\ &+\sum_{m=s+1}^{\infty}\frac{1}{l}\Big\{C'_m G\dot{e}k_{2n+1}(\xi, -q_1')se_{2n+1}(\eta, -q_1') - D'_m Fek_{2n+1}(\xi, -q_2') \\ &\qquad \cdot c\dot{e}_{2n+1}(\eta, -q_2) - \frac{4u_0 k\cosh\xi}{m\pi\xi^2{}_m}\left(\frac{\omega}{\omega_g}\right)^2\sin\eta\Big\}\sin\frac{m\pi z}{2H}\cdot e^{i\omega t} \end{aligned}\right\}\cdots\cdots(24)$$

$$\left.\begin{aligned} u_\eta = &\sum_{m=1,3,5,\cdots}^{s} \frac{1}{l}\Big\{C_m \overset{(2)}{Ne}_{2n+1}(\xi, q_1)s\dot{e}_{2n+1}(\eta, q_1) + D_m \dot{\overset{(2)}{Me}}_{2n+1}(\xi, q_2)ce_{2n+1}(\eta, q_2) \\ &\qquad - \frac{4u_0 k\sinh\xi}{m\pi\xi^2{}_m}\left(\frac{\omega}{\omega_g}\right)^2\cos\eta\Big\}\sin\frac{m\pi z}{2H}\cdot e^{i\omega t} \\ &+\sum_{m=s+1}^{\infty}\frac{1}{l}\Big\{C'_m Gek_{2n+1}(\xi, -q_1')s\dot{e}_{2n+1}(\eta, -q_1') + D'_m Fek_{2n+1}(\xi, -q_2') \\ &\qquad \cdot ce_{2n+1}(\eta, -q_2') - \frac{4u_0 k\sinh\xi}{m\pi\xi^2{}_m}\left(\frac{\omega}{\omega_g}\right)^2\cos\eta\Big\}\sin\frac{m\pi z}{2H}\cdot e^{i\omega t} \end{aligned}\right\}\cdots(25)$$

in which s is the maximum positive odd integer which satisfies the condition $\xi^2{}_m > 0$. C_m, D_m, C'_m and D'_m are the integration constants. $\dot{\overset{(2)}{Me}}_{2n+1}(\xi, q_2)$, $\dot{\overset{(2)}{Ne}}_{2n+1}(\xi, q_1)$ denote the derivatives of $\overset{(2)}{Me}_{2n+1}(\xi, q_2)$, $\overset{(2)}{Ne}_{2n+1}(\xi, q_1)$ with respect to ξ. Needless to say u_ξ, u_η in eqs. (24), (25) satisfy all boundary conditions in eq. (8) except for the condition (iii) by which the undetermined coefficients C_m, D_m are to be decided. Since the analysis for the case where the arguments q_1, q_2 are negative can be made in a similar manner only by substituting $Fek_{2n+1}(\xi, -q_2)$, $Gek_{2n+1}(\xi, -q_1)$ for $\overset{(2)}{Me}_{2n+1}(\xi, -q_2)$, $\overset{(2)}{Ne}_{2n+1}(\xi, -q_1)$, we shall henceforth represent the modified Mathieu functions by $Me^{(2)}_{2n+1}(\xi, q)$, $Ne^{(2)}_{2n+1}(\xi, q)$.

(2) Frequency Response for the Rocking Vibration of a Rigid Foundation Structure

The vibration model dealt with in this section would correspond to caissons or well foundations. For the vibration in the direction of the minor axis, the foundation structure is assumed to be a rigid elliptic cylinder which rotates around the center line of the bottom, or the x axis, with angular amplitude φ.

If u_p is the horizontal displacement of the elliptic cylinder in the direction of the minor axis, then the displacements $u_{p,\xi}$, $u_{p,\eta}$ in the directions of ξ, η are written, respectively, as follows:

$$u_{p,\xi}=\varphi_0 z\sin\theta\, e^{i\omega t}=\frac{8\varphi_0 H}{\pi^2}\frac{a\sin\eta}{l}\sum_{m=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\sin\frac{m\pi z}{2H}e^{i\omega t} \qquad \cdots\cdots(26)$$

$$u_{p,\eta}=\varphi_0 z\cos\theta\, e^{i\omega t}=\frac{8\varphi_0 H}{\pi^2}\frac{a\cos\eta}{l}\sum_{m=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\sin\frac{m\pi z}{2H}e^{i\omega t} \qquad \cdots\cdots(27)$$

Putting eqs. (26), (27) equal to eqs. (24), (25) by use of the boundary conditions (iii) of eq. (8) gives

$$C_m\dot{N}e^{(2)}_{2n+1}(\xi_0, q_1)se_{2n+1}(\eta, q_1)-D_m Me^{(2)}_{2n+1}(\xi_0, q_2)c\dot{e}_{2n+1}(\eta, q_2)$$

$$=\left\{\frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2+\frac{8\varphi_0 H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\right\}a\sin\eta \qquad \cdots\cdots\cdots(28)$$

$$C_m Ne^{(2)}_{2n+1}(\xi_0, q_1)s\dot{e}_{2n+1}(\eta, q_1)+D_m Me^{(2)}_{2n+1}(\xi, q_2)ce_{2n+1}(\eta, q_2)$$

$$=\left\{\frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2+\frac{8\varphi_0 H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\right\}b\cos\eta \qquad \cdots\cdots\cdots(29)$$

By virtue of the orthogonality of the Mathieu functions $se_{2n+1}(\eta, q_1)$ and $ce_{2n+1}(\eta, q_2)$ in the domain of $(0, 2\pi)$, we have the following expressions for C_m and D_m:

$$\left.\begin{aligned}C_m=&\left\{\frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2+\frac{8\varphi_0 H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\right\}\Big[aB_1^{(1)}\sum_{r=0}^{\infty}(A^{(2)}_{2r+1})^2\dot{M}e^{(2)}_1(\xi_0, q_2)\\&-bA^{(1)}_1\sum_{r=0}^{\infty}(2r+1)A^{(1)}_{2r+1}B^{(1)}_{2r+1}Me^{(2)}_1(\xi, q_2)\Big]\Big/\Big[\sum_{r=0}^{\infty}(A^{(1)}_{2r+1})^2\sum_{r=0}^{\infty}(B^{(1)}_{2r+1})^2\dot{M}e^{(2)}_1(\xi_0, q_2)\\&\cdot\dot{N}e^{(2)}_1(\xi_0, q_1)-\Big(\sum_{r=0}^{\infty}(2r+1)A^{(1)}_{2r+1}B^{(1)}_{2r+1}\Big)^2Me^{(2)}_1(\xi_0, q_2)Ne^{(2)}_1(\xi_0, q_1)\Big],\quad (n=0)\\&=0,\quad (n\neq0)\end{aligned}\right\}\cdots(30)$$

$$\left.\begin{aligned}D_m=&\left\{\frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2+\frac{8\varphi_0 H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\right\}\Big[bA_1^{(1)}\sum_{r=0}^{\infty}(B^{(1)}_{2r+1})^2\dot{N}e^{(2)}_1(\xi_0, q_1)\\&-aB^{(1)}_1\sum_{r=0}^{\infty}(2r+1)A^{(1)}_{2r+1}B^{(1)}_{2r+1}Ne^{(2)}_1(\xi_0, q_1)\Big]\Big/\Big[\sum_{r=0}^{\infty}(A^{(1)}_{2r+1})^2\sum_{r=0}^{\infty}(B^{(1)}_{2r+1})^2\dot{M}e^{(2)}_1(\xi_0, q_2)\\&\cdot\dot{N}e^{(2)}_1(\xi_0, q_1)-\Big(\sum_{r=0}^{\infty}(2r+1)A^{(1)}_{2r+1}B^{(1)}_{2r+1}\Big)^2Me^{(2)}_1(\xi_0, q_2)Ne^{(2)}_1(\xi_0, q_1),\quad (n=0)\\&=0,\quad (n\neq0)\end{aligned}\right\}\cdots(31)$$

When $\xi^2_m<0$, the undetermined coefficients C_m, D_m are obtained only by replacing the complex functions $Me^{(2)}_1(\xi_0, -q_2)$, $Ne^{(2)}_1(\xi_0, -q_1)$ by the real functions

$Fek_1(\xi_0, -q_2)$, $Gek_1(\xi_0, -q_1)$, respectively. Then the relations between stress and displacement are given in the elliptic cylindrical coordinates as follows:

$$\sigma_\xi=\lambda\left\{\frac{\partial(lu_\xi)}{l^2\partial\xi}+\frac{\partial(lu_\eta)}{l^2\partial\eta}\right\}+2\mu\left\{\frac{\partial u_\xi}{l\partial\xi}+\frac{u_\eta}{l^3}\frac{\partial l}{\partial\eta}\right\} \quad \cdots\cdots(32)$$

$$\tau_{\xi\eta}=\mu\left\{\frac{\partial}{\partial\xi}\left(\frac{u_\eta}{l}\right)+\frac{\partial}{\partial\eta}\left(\frac{u_\xi}{l}\right)\right\} \quad \cdots\cdots(33)$$

When the bedrock vibrates in the direction of the minor axis, earth pressure $p(z)$ on the elliptic cylinder per unit length is written by neglecting the complicated processes of induction as

$$p(z)=\oint_{\xi_0}(\sigma_\xi\cdot\sin\theta+\tau_{\xi\eta}\cdot\cos\theta)\,ds=\int_0^{2\pi}(a\sin\eta\cdot\sigma_\xi+b\cos\eta\cdot\tau_{\xi\eta})\,d\eta$$

$$=-\sum_{m=1,3,5,\cdots}^{\infty}\rho\pi\omega_g^2\xi_m^2\left\{C_maB_1^{(1)}Ne_1^{(2)}(\xi_0,q_1)+D_mbA_1^{(1)}Me_1^{(2)}(\xi_0,q_2)\right\}\sin\frac{m\pi z}{2H}e^{i\omega t}$$

$$=-\sum_{m=1,3,5,\cdots}^{\infty}\rho\pi ab\Omega_m\left\{\frac{8\varphi_0H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}+\frac{4u_0}{m\pi\xi_m^2}\left(\frac{\omega}{\omega_g}\right)^2\right\}\xi_m^2\omega_m^2\sin\frac{m\pi z}{2H}e^{i\omega t} \quad \cdots\cdots(34)$$

Thus we get the following overturning moment M around the center line of the bottom of the elliptic cylinder:

$$M=\int_0^H p(z)z\,dz$$

$$=-\sum_{m=1,3,5,\cdots}^{\infty}\rho\pi\omega_g^2\xi_m^2\left(\frac{2H}{m\pi}\right)^2(-1)^{\frac{m-1}{2}}\left\{C_maB_1^{(1)}Ne_1^{(2)}(\xi_0,q_1)+D_mbA_1^{(1)}Me_1^{(2)}(\xi_0,q_2)\right\}e^{i\omega t}$$

$$=-\sum_{m=1,3,5,\cdots}^{\infty}\rho\pi ab\left(\frac{2H}{m\pi}\right)^2(-1)^{\frac{m-1}{2}}\Omega_m\left\{\frac{8\varphi_0H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}+\frac{4u_0}{m\pi\xi_m^2}\left(\frac{\omega}{\omega_g}\right)^2\right\}\xi^3_m\omega^2_g e^{i\omega t} \quad \cdots\cdots(35)$$

in which

$$\Omega_m=\Bigg[\frac{a}{b}(B_1^{(1)})^2\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_2)}{Me_1^{(2)}(\xi_0,q_2)}+\frac{b}{a}(A_1^{(1)})^2\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\frac{\dot{N}e_1^{(2)}(\xi_0,q_1)}{Ne_1^{(2)}(\xi_0,q_1)}$$
$$-2A_1^{(1)}B_1^{(1)}\sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(1)}B_{2r+1}^{(1)}\Bigg]\Bigg/\Bigg[\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_2)}{Me_1^{(2)}(\xi_0,q_2)}$$
$$\cdot\frac{\dot{N}e_1^{(2)}(\xi_0,q_1)}{Ne_1^{(2)}(\xi_0,q_1)}-\left(\sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(1)}B_{2r+1}^{(1)}\right)^2\Bigg] \quad \cdots\cdots(36)$$

It is clear that eqs. (34), (35) and (36) for the elliptic cylinder are entirely similar to the solutions for circular cylinders. Namely, putting $a=b$ in eq. (36), we obtain

$$A_{2r+1}^{(1)}=B_{2r+1}^{(1)}=\begin{cases}1 & (r=0)\\ 0 & (r\neq0)\end{cases}$$

Thus replacing the Mathieu functions in eqs. (35), (36) with the Hankel functions, we get the following solutions for the circular cylinder:

$$M=-\sum_{m=1,3,5,\cdots}^{\infty}\rho\pi a^2\left(\frac{2H}{m\pi}\right)^2(-1)^{\frac{m-1}{2}}\Omega_m\left\{\frac{8\varphi_0H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}+\frac{4u_0}{m\pi\xi_m^2}\left(\frac{\omega}{\omega_g}\right)^2\right\}\xi^2_m\omega^2_g e^{i\omega t} \quad \cdots\cdots(37)$$

and

$$\Omega_m=\left[a\frac{\dot{H}_1^{(2)}(\alpha'_ma)}{H_1^{(2)}(\alpha'_ma)}+a\frac{\dot{H}_1^{(2)}(\beta'_ma)}{H_1^{(2)}(\beta'_ma)}-2\right]\Bigg/\left[a^2\frac{\dot{H}_1^{(2)}(\alpha'_ma)}{H_1^{(2)}(\alpha'_ma)}\frac{\dot{H}_1^{(2)}(\beta'_ma)}{H_1^{(2)}(\beta'_ma)}-1\right] \quad \cdots\cdots(38)$$

in which a is the radius of the circular cylinder, and

$$\alpha'_m a=\frac{\pi}{2}\frac{a}{H}\frac{v_t}{v_l}\xi_m,\ \beta'_m a=\frac{\pi}{2}\frac{a}{H}\xi_m,\ \xi_m=\sqrt{\left(\frac{\omega}{\omega_g}\right)^2-m^2},\ (m=1,3,5,\cdots\cdots) \quad\cdots(39)$$

Besides, for the case of $b/a=0$, which corresponds to a thin plate with the width $2a$, we have

$$M=-\sum_{m=1,3,5,\cdots}^{\infty}\rho\pi a^2\left(\frac{2H}{m\pi}\right)^2(-1)^{\frac{m-1}{2}}\Omega_m\left\{\frac{8\varphi_0 H}{\pi^2}\frac{(-1)^{\frac{m-1}{2}}}{m^2}+\frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2\right\}\xi^2_m\omega^2_g e^{i\omega t} \quad\cdots\cdots\cdots(40)$$

and

$$\Omega_m=(B_1^{(1)})^2\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_2)}{Me_1^{(2)}(\xi_0,q_2)}\Big/\Big[\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_2)}{Me_1^{(2)}(\xi_0,q_2)}\cdot\frac{\dot{N}e_1^{(2)}(\xi_0,q_1)}{Ne_1^{(2)}(\xi_0,q_1)}-\left(\sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(1)}B_{2r+1}^{(1)}\right)^2\Big] \quad\cdots\cdots\cdots(41)$$

On the other hand, the equation of the rocking motion of the elliptic cylinder is written as

$$I\ddot{\varphi}+k_r\varphi=m_0u_0\omega^2H_g e^{i\omega t}+M \quad\cdots\cdots\cdots(42)$$

in which I is the moment of inertia around the center line at the bottom of the elliptic cylinder. Substitution $\varphi=\varphi_0 e^{i\omega t}$ into eq. (42) yields

$$\varphi_0=-\frac{m_0H_g-\dfrac{16ab\rho H^3}{\pi^2}\displaystyle\sum_{m=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^3}\Omega_m}{k_r-I\omega^2+\dfrac{32ab\rho H^3}{\pi^3}\omega_g^2\displaystyle\sum_{m=1,3,5,\cdots}^{\infty}\xi^2_m\frac{\Omega_m}{m^4}} \quad\cdots\cdots\cdots(43)$$

We adopt the notations used by Prof. Tajimi in the analysis on circular cylinders to facilitate the comparison. For $\omega=0$, the third term in the denominator of eq. (43) corresponds to the static horizontal spring constant. If we let χ denote the ratio of this static horizontal spring constant to that for the rocking vibration, χ is shown as follows:

$$\frac{32ab\rho H^3}{\pi^3}\omega_g^2\sum_{m=1,3,5,\cdots}^{\infty}\xi^2_m\frac{\Omega_m}{m^4}=-\frac{32ab\rho H^3}{\pi^3}\omega_g^2\sum_{m=1,3,5,\cdots}^{\infty}\frac{\Omega_{ms}}{m^2}=\chi k_r \quad\cdots\cdots\cdots(44)$$

Now set

$$-\frac{\displaystyle\sum_{m=1,3,5,\cdots}^{\infty}\xi_m^2\frac{\Omega_m}{m^4}}{\displaystyle\sum_{m=1,3,5,\cdots}^{\infty}\frac{\Omega_{ms}}{m^2}}=f_1+if_2,\quad \frac{\displaystyle\sum_{m=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^3}\Omega_m}{\displaystyle\sum_{m=1,3,5,\cdots}^{\infty}\frac{\Omega_{ms}}{m^2}}=g_1+ig_2 \quad\cdots\cdots\cdots(45)$$

in which f_1, f_2, g_1 and g_2 are real values. Then from eq. (43), f_1+if_2 in eq. (45) represents the dynamic spring effect due to soil reaction and g_1+ig_2 shows the dynamic effect of the earth pressure or soil reaction. By use of an expression $\kappa(z)$ which is the horizontal amplification factor of foundation structures at an arbitrary height z from the bedrock, we obtain

$$\kappa(z)=\left|\frac{\varphi_0 z+u_0}{u_0}\right|=\left|1+\frac{\dfrac{H_g z}{i_0^2}\left(\dfrac{\omega_g}{\omega_s}\right)^2+\dfrac{\pi}{2}\dfrac{z}{H}\chi(g_1+ig_2)}{1-\left(\dfrac{\omega}{\omega_s}\right)^2+\chi(f_1+if_2)}\left(\frac{\omega}{\omega_g}\right)^2\right| \quad\cdots\cdots\cdots(46)$$

in which $i_0=\sqrt{I/m}$, $\omega_s=\sqrt{k_r/H}$, $m_0=\rho_p\pi abH_s$, $I=m_0(b^2/4+H_s^2/3)$.

For the case of $b/a=0$, the amplification factor $\kappa(z)$ becomes

$$\kappa(z)=\left|1-\frac{\pi}{2}\cdot\frac{z}{H}\frac{\sum_{m=1,3,5,\cdots}^{\infty}\frac{(-1)^{\frac{m-1}{2}}}{m^2}\Omega_m}{\sum_{m=1,3,5,\cdots}^{\infty}\xi_m^2\frac{\Omega_m}{m^4}}\left(\frac{\omega}{\omega_g}\right)^2\right|$$

$$=\left|1+\frac{\pi}{2}\cdot\frac{z}{H}\frac{g_1+ig_2}{f_1+if_2}\left(\frac{\omega}{\omega_g}\right)^2\right| \qquad \cdots\cdots\cdots(47)$$

where Ω_m must be computed by use of eq. (41).

Thus we have analyzed the frequency response of an elliptic cylinder vibrating in the direction of the minor axis. In the same way, the amplification factor $\kappa(z)$ in eq. (46) applies to the vibration of an elliptic cylinder in the direction of the major axis, though Ω_m must be calculated from

$$\Omega_m=\left[\frac{a}{b}(B_1^{(1)})^2\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_1)}{Me_1^{(2)}(\xi_0,q_1)}+\frac{b}{a}(A_1^{(1)})^2\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\frac{\dot{N}e_1^{(2)}(\xi_0,q_2)}{Ne_1^{(2)}(\xi_0,q_2)}\right.$$
$$\left.-2A_1^{(1)}B_1^{(1)}\sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(1)}B_{2r+1}^{(1)}\right]\Big/\left[\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_1)}{Me_1^{(2)}(\xi_0,q_1)}\right.$$
$$\left.\cdot\frac{\dot{N}e_1^{(2)}(\xi_0,q_2)}{Ne_1^{(2)}(\xi_0,q_2)}-\left(\sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(2)}B_{2r+1}^{(1)}\right)^2\right] \qquad \cdots\cdots\cdots(48)$$

in which the coefficients $A_{2r+1}^{(1)}$, $B_{2r+1}^{(1)}$ are functions of q_1, m and q_2, m, respectively. The arguments q_1, q_2 are defined as in eq. (23).

Besides for the case of $b/a=0$, the amplification factor $\kappa(z)$ takes the same form as eq. (47), except that Ω_m is represented by

$$\Omega_m=(B_1^{(1)})^2\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_1)}{Me_1^{(2)}(\xi_0,q_1)}\Big/\left[\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2\frac{\dot{M}e_1^{(2)}(\xi_0,q_1)}{Me_1^{(2)}(\xi_0,q_1)}\right.$$
$$\left.\cdot\frac{\dot{N}e_1^{(2)}(\xi_0,q_2)}{Ne_1^{(2)}(\xi_0,q_2)}-\left(\sum_{r=0}^{\infty}(2r+1)A_{2r+1}^{(1)}B_{2r+1}^{(1)}\right)^2\right] \qquad \cdots\cdots\cdots(49)$$

in which the coefficients $A_{2r+1}^{(1)}$, $B_{2r+1}^{(1)}$ and the arguments q_1, q_2 must be treated in the same manner as in the analysis of an elliptic cylinder vibrating in the direction of the minor axis.

With regard to the static spring constant k_r for the rocking vibration of a rigid body on the surface of a semi-infinite elastic medium, Prof. Timoshenko has presented the following relation between k_r and other physical constants:

$$k_r=\frac{2\rho_b^*v_t^{*2}I_0^*}{\beta(1-\nu)\sqrt{A}} \qquad \cdots\cdots\cdots(50)$$

where ρ_b^* is density of elastic medium, v_t^* velocity of transverse wave in elastic medium, I_0^* geometrical moment of inertia at the bottom section of the rigid body and A is the bottom area of the rigid body. For the case of vibration in the direction of the minor axis, we have $A=\pi ab$, $I_0=\pi ab^3/4$.

Now supposing that C_1^*, C_2^* are the coefficients including these physical constants when neither the densities of the surface layer and the bedrock nor the velocity of the transverse wave in the surface layer are variable, the parameters χ and ω_s/ω_g are written as

$$\chi=C_1^*\frac{H}{b}\sqrt{\frac{a}{b}}\sum_{m=1,3,5,\cdots}^{\infty}\frac{\Omega_{ms}}{m^2} \qquad \cdots\cdots\cdots(51)$$

$$\frac{\omega_s}{\omega_g}=C_2^*\sqrt[4]{\frac{b}{a}}\Big/\sqrt{\frac{b}{4H}+\frac{H}{3b}} \qquad \cdots\cdots\cdots(52)$$

for the case of vibration in the direction of the minor axis, and

$$\chi=C_1^*\frac{H}{a}\sqrt{\frac{b}{a}}\sum_{m=1,3,5,\cdots}^{\infty}\frac{\Omega_{ms}}{m^3} \qquad \cdots\cdots\cdots(53)$$

$$\frac{\omega_s}{\omega_g}=C_2^*\sqrt[4]{\frac{a}{b}}\Big/\sqrt{\frac{a}{4H}+\frac{H}{3a}} \qquad \cdots\cdots\cdots(54)$$

for the case of vibration in the direction of the major axis, in which χ is the same as in eq. (44).

(3) Frequency Response for the Elastic Vibration of a Flexible Foundation Structure

Pile foundations would correspond to this case, for which the following assumptions are considered. (1) The foundation structure is a flexible cylinder with an elliptic cross section. (2) The superstructure is not considered. (3) The bottom of the cylinder is rigidly fixed to the bedrock as shown in Fig. 2.

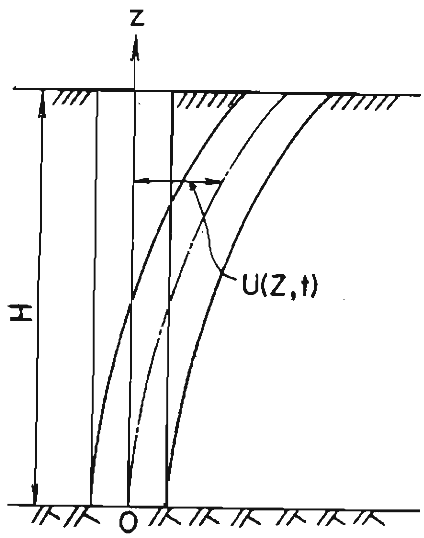

Fig. 2 Foundation-Structure Systems Considered

Let $u_p(z,t)$ denote the relative displacement of this flexible cylinder to the bedrock, then the equation of motion of the elliptic cylinder is written as follows:

$$\rho_p{}^*\pi ab\frac{\partial^2 u_p}{\partial t^2}+EI_0\frac{\partial^4 u_p}{\partial z^4}=-\rho_p{}^*\pi ab\cdot\frac{\partial^2 u_g}{\partial t^2}+p(z) \qquad \cdots\cdots\cdots(55)$$

in which $p(z)$ is the soil reaction due to earth pressure acting on a unit length of elliptic cylinder analogous to that in eq. (34). The deflection $U_p(z)$ of the elliptic cylinder can be expanded in the series of the characteristic functions $\eta_\mu(k_\mu z)$ of a cantilever.

$$u_p=U_p(z)e^{i\omega t}=\sum_{\mu=1}^{\infty}A_\mu\eta_\mu(k_\mu z)e^{i\omega t} \qquad \cdots\cdots(56)$$

where A_μ is the undetermined coefficient and $k_\mu H$ is the characteristic value. Thus for the elastic vibration in the direction of the minor axis the following simultaneous equations on the undetermind coefficients C_m, D_m are obtained by consulting the condition of continuity between the displacement $u_p(z,t)$ of the elliptic cylinder in eq. (56) and the displacements u_ξ, u_η of the surface layer in eqs. (24), (25) at $\xi=\xi_0$.

$$u_\xi=u_p\sin\theta=u_p\frac{a}{l}\sin\eta=\frac{a}{l}\sin\eta\sum_{\mu=1}^{\infty}A_\mu\eta_\mu(k_\mu z)e^{i\omega t}$$

$$=\sum_{m=1,3,5,\cdots}^{\infty}\frac{1}{l}\Big\{C_m\dot{N}e_1^{(2)}(\xi_0,q_1)se_1(\eta,q_1)-D_mMe_1^{(2)}(\xi_0,q_2)c\dot{e}_1(\eta,q_2)$$

$$-\frac{4u_0 k \cosh\xi_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2 \sin\eta\Big\} \sin\frac{m\pi z}{2H} e^{i\omega t} \qquad \cdots\cdots\cdots(57)$$

$$u_\eta = u_p \cos\theta = u_p \frac{b}{l}\cos\eta = \frac{b}{l}\cos\eta \sum_{\mu=1}^{\infty} A_\mu \eta_\mu (k_\mu z) e^{i\omega t}$$

$$= \sum_{m=1,3,5,\cdots}^{\infty} \frac{1}{l}\Big\{ C_m Ne_1^{(2)}(\xi_0, q_1) s\dot{e}_1(\eta, q_1) + D_m \dot{M}e_1^{(2)}(\xi_0, q_2) ce_1(\eta, q_2)$$

$$-\frac{4u_0 k \sinh\xi_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2 \cos\eta\Big\} \sin\frac{m\pi z}{2H} e^{i\omega t} \qquad \cdots\cdots\cdots(58)$$

By virtue of the orthogonality of $\sin(n\pi z/2H)$ $(n=1, 3, 5, \cdots\cdots)$ in the region of $(0, H)$, the following expressions for C_m and D_m are obtained:

$$C_m = \Big\{ 2\sum_{\mu=1}^{\infty} A_\mu G_\mu + \frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2 \Big\} \Big[aB_1^{(1)} \sum_{r=0}^{\infty} (A_{2r+1}^{(1)})^2 \dot{M}e_1^{(2)}(\xi_0, q_2) - bA_1^{(1)} \sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(1)}$$

$$\cdot B_{2r+1}^{(1)} Me_1^{(2)}(\xi_0, q_2) \Big] \Big/ \Big[\sum_{r=0}^{\infty} (A_{2r+1}^{(1)})^2 \sum_{r=0}^{\infty} (B_{2r+1}^{(1)})^2 \dot{M}e_1^{(2)}(\xi_0, q_2) \dot{N}e_1^{(2)}(\xi_0, q_1) - \Big(\sum_{r=0}^{\infty} (2r+1)$$

$$\cdot A_{2r+1}^{(1)} B_{2r+1}^{(1)} \Big)^2 Me_1^{(2)}(\xi_0, q_2) Ne_1^{(2)}(\xi_0, q_1) \Big] \qquad \cdots\cdots\cdots(59)$$

$$D_m = \Big\{ 2\sum_{\mu=1}^{\infty} A_\mu G_\mu + \frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2 \Big\} \Big[bA_1^{(1)} \sum_{r=0}^{\infty} (B_{2r+1}^{(1)})^2 \dot{N}e_1^{(2)}(\xi_0, q_1) - aB_1^{(1)} \sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(1)}$$

$$\cdot B_{2r+1}^{(1)} Ne_1^{(2)}(\xi_0, q_1) \Big] \Big/ \Big[\sum_{r=0}^{\infty} (A_{2r+1}^{(1)})^2 \sum_{r=0}^{\infty} (B_{2r+1}^{(1)})^2 \dot{M}e_1^{(2)}(\xi_0, q_2) \dot{N}e_1^{(2)}(\xi_0, q_1) - \Big(\sum_{r=0}^{\infty} (2r+1)$$

$$\cdot A_{2r+1}^{(1)} B_{2r+1}^{(1)} \Big)^2 Me_1^{(2)}(\xi_0, q_2) Ne_1^{(2)}(\xi_0, q_1) \Big] \qquad \cdots\cdots\cdots(60)$$

where

$$G_\mu = \frac{1}{H}\int_0^H \eta_\mu(k_\mu z) \sin\frac{m\pi z}{2H} dz, \quad (m=1, 3, 5, \cdots\cdots) \qquad \cdots\cdots\cdots(61)$$

Thus we get the soil reaction due to earth pressure acting on the elliptic cylinder from eq. (34) as follows:

$$p(z) = -\sum_{m=1,3,5,\cdots}^{\infty} \rho\pi\omega_g^2 \xi^2_m \Big\{ C_m a B_1^{(1)} Ne_1^{(2)}(\xi_0, q_1) + D_m b A_1^{(1)} Me_1^{(2)}(\xi_0, q_2) \Big\} \sin\frac{m\pi z}{2H} e^{i\omega t}$$

$$= -\sum_{m=1,3,5,\cdots}^{\infty} \rho\pi ab \omega_g^2 \xi^2_m \Omega_m \Big\{ 2\sum_{\mu=1}^{\infty} A_\mu G_\mu + \frac{4u_0}{m\pi\xi^2_m}\left(\frac{\omega}{\omega_g}\right)^2 \Big\} \sin\frac{m\pi z}{2H} e^{i\omega t} \qquad \cdots\cdots\cdots(62)$$

in which

$$\Omega_m = \Big[\frac{a}{b}(B_1^{(1)})^2 \sum_{r=0}^{\infty} (A_{2r+1}^{(1)})^2 \frac{\dot{M}e_1^{(2)}(\xi_0, q_2)}{Me_1^{(2)}(\xi_0, q_2)} + \frac{b}{a}(A_1^{(1)})^2 \sum_{r=0}^{\infty} (B_{2r+1}^{(1)})^2 \frac{\dot{N}e_1^{(2)}(\xi_0, q_1)}{Ne_1^{(2)}(\xi_0, q_1)}$$

$$-2A_1^{(1)} B_1^{(1)} \sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(1)} B_{2r+1}^{(1)} \Big] \Big/ \Big[\sum_{r=0}^{\infty} (A_{2r+1}^{(1)})^2 \sum_{r=0}^{\infty} (B_{2r+1}^{(1)})^2 \frac{\dot{M}e_1^{(2)}(\xi_0, q_2)}{Me_1^{(2)}(\xi_0, q_1)}$$

$$\cdot \frac{\dot{N}e_1^{(2)}(\xi_0, q_1)}{Ne_1^{(2)}(\xi_0, q_1)} - \Big(\sum_{r=0}^{\infty} (2r+1) A_{2r+1}^{(1)} B_{2r+1}^{(1)} \Big)^2 \Big] \qquad \cdots\cdots\cdots(63)$$

and

$$q_1=\left(\frac{\pi}{4}\right)^2\left(\frac{k}{H}\right)^2\left(\frac{v_t}{v_l}\right)^2\xi^2{}_m,\ q_2=\left(\frac{\pi}{4}\right)^2\left(\frac{k}{H}\right)^2\xi^2{}_m,\ \xi^2{}_m=\left(\frac{\omega}{\omega_g}\right)^2-m^2,\ (m=1,3,5,\cdots)$$

The coefficients $A^{(1)}_{2r+1}$ and $B^{(1)}_{2r+1}$ peculiar to the Mathieu functions are the functions of q_2, m and q_1, m, respectively. On the other hand, the following relation is obtained with the aid of the natural frequency ω_μ $(\mu=1,2,3,\cdots\cdots)$ of the elliptic cylinder in the air.

$$EI_0\frac{\partial^4\eta_\mu(k_\mu z)}{\partial z^4}=\rho_p{}^*\pi ab\omega^2{}_\mu\,\eta_\mu(k_\mu z) \qquad\cdots\cdots\cdots(64)$$

Now the motion u_g of the bedrock is assumed to be

$$u_g=u_0e^{i\omega t} \qquad\cdots\cdots\cdots(65)$$

as in the case of the rocking vibration of the rigid elliptic cylinder. Substituting eqs. (62)~(65) into eq. (55) leads to

$$\rho_p{}^*\pi ab\sum_{\mu=1}^{\infty}(\omega^2{}_\mu-\omega^2)A_\mu\eta_\mu(k_\mu z)$$

$$=\rho_p{}^*\pi abu_0\omega^2-\sum_{m=1,3,5,\cdots}^{\infty}\rho\pi ab\xi^2{}_m\omega_g{}^2\Omega_m\left\{2\sum_{\mu=1}^{\infty}A_\mu G_\mu+\frac{4u_0}{m\pi^2\xi_m{}^2}\left(\frac{\omega}{\omega_g}\right)^2\right\}\sin\frac{m\pi z}{2H} \qquad\cdots\cdots(66)$$

By virtue of the orthogonality of the characteristic functions $\eta_\nu(k_\nu z)$, the above equation yields

$$\left\{\left(\frac{\omega_\nu}{\omega}\right)^2-1\right\}A_\nu\Phi_\nu=u_0(D_\nu+F_\nu)+\sum_{\mu=1}^{\infty}A_\mu\Theta_{\nu\mu},\quad(\nu=1,2,3,\cdots) \qquad\cdots\cdots\cdots(67)$$

in which

$$\left.\begin{aligned}
\Phi_\nu&=\frac{1}{H}\int_0^H\left\{\eta_\nu(k_\nu z)\right\}^2dz\\
F_\nu&=\frac{1}{H}\int_0^H\eta_\nu(k_\nu z)dz\\
D_\nu&=-\frac{\rho}{\rho_p{}^*}\sum_{m=1,3,5,\cdots}^{\infty}\frac{4}{m\pi}\Omega_mG_\nu\\
\Theta_{\nu\mu}&=-\frac{\rho}{\rho_p{}^*}\sum_{m=1,3,5,\cdots}^{\infty}2\left(\frac{\omega_g}{\omega}\right)^2\xi^2{}_m\Omega_mG_\nu G_\mu\\
G_\nu&=\frac{1}{H}\int_0^H\eta_\nu(k_\nu z)\,\sin\frac{m\pi z}{2H}dz,\quad(m=1,3,5,\cdots\cdots)
\end{aligned}\right\} \qquad\cdots\cdots\cdots\cdots\cdots\cdots(68)$$

Eq. (67) is the set of equations concerning the undetermined coefficients A_ν $(\nu=1,2,3,\cdots\cdots)$. The normalized characteristic function $\eta_\nu(k_\nu z)$ of a cantilever is given in the following form:

$$\eta_\nu(k_\nu z)=\beta_\nu\left\{\sinh k_\nu z-\sin k_\nu z-\frac{\sinh k_\nu H+\sin k_\nu H}{\cosh k_\nu H+\cos k_\nu H}(\cosh k_\nu z-\cos k_\nu z)\right\} \qquad\cdots\cdots\cdots(69)$$

in which β_ν is a constant. The coefficient β_ν is determined from the normalization condition

$$\frac{1}{H}\int_0^H\left\{\eta_\nu(k_\nu z)\right\}^2dz=1 \qquad\cdots\cdots\cdots(70)$$

We consequently get

$$\beta_\nu=\frac{\cosh k_\nu H+\cos k_\nu H}{\sinh k_\nu H+\sin k_\nu H} \qquad\cdots\cdots\cdots(71)$$

When $k_\nu H\neq\pi/2$, carrying out the integrations in eq. (68) gives

$$\left.\begin{aligned}
&\Phi_\nu=1\\
&F_\nu=-\frac{2}{kH}\cdot\beta_\nu\\
&D_\nu=-\frac{\rho}{\rho_p{}^*}\sum_{m=1,3,5,\cdots}^{\infty}\frac{4}{m\pi}\Omega_m G_\nu\\
&\Theta_{\nu\mu}=-\frac{\rho}{\rho_p{}^*}\sum_{m=1,3,5,\cdots}^{\infty}2\left(\frac{\omega}{\omega_g}\right)^2\xi^2{}_m\Omega_m G_\nu G_\mu\\
&G_\nu=(-1)^{\frac{m-1}{2}}\left[\frac{\left(\frac{m\pi}{2}\beta_\nu-k_\nu H\right)\sinh k_\nu H}{(k_\nu H)^2+\left(\frac{m\pi}{2}\right)^2}+\frac{k_\nu H\ (\sin k_\nu H+\beta_\nu\cos k_\nu H)}{(k_\nu H)^2-\left(\frac{m\pi}{2}\right)^2}\right]\\
&\qquad-\frac{k_\nu H\left[\beta_\nu\cdot(k_\nu H)^2+2\cdot\frac{m\pi}{2}\cdot k_\nu H-\beta_\nu\cdot\left(\frac{m\pi}{2}\right)^2\right]}{(k_\nu H)^4-\left(\frac{m\pi}{2}\right)^4}
\end{aligned}\right\}\quad\cdots\cdots\cdots(72)$$

If $k_\nu H=m\pi/2$, G_ν is represented by

$$G_\nu=\frac{1}{2k_\nu H}\left[(-1)^{\frac{m-1}{2}}(\beta_\nu-1)\sinh k_\nu H-\beta_\nu(1+k_\nu H)\right]\quad\cdots\cdots\cdots(73)$$

where the characteristic values $k_\nu H$ ($\nu=1,2,3,\cdots\cdots$) are given as follows:

$k_1H=1.875,\ \ k_2H=4.694,\ \ k_3H=7.855,\ \cdots\cdots$ $\cdots\cdots\cdots(74)$

The foregoing analysis was made for the case of elastic vibration in the direction of the minor axis. The method of analysis for the case of elastic vibration in the direction of the major axis is identical to the above-mentioned case, where Ω_m is the same value as in eq. (48) and the arguments q_1, q_2, in eq. (23).

Consequently, the amplification factor $\kappa(z)$ for the horizontal displacement at an arbitrary height z from the bedrock is represented by

$$\kappa(z)=\left|\frac{U_p(z)+u_0}{u_0}\right|=\left|1+\frac{1}{u_0}\sum_{\mu=1}^{\infty}A_\mu\,\eta_\mu(k_\mu z)\right|\quad\cdots\cdots\cdots(75)$$

For a special case where the foundation structure is a circular cylinder, the same treatment can be made as in the case of an elliptic cylinder only by replacing the Mathieu functions by Hankel functions.

Besides, for the case of $b/a=0$ which corresponds to a thin flexible plate, the simultaneous equations (67) for the direction of the minor axis become

$$-\sum_{\mu=1}^{\infty}A_\mu\Theta_{\nu\mu}=u_0D_\nu,\quad(\nu=1,2,3,\cdots\cdots)\quad\cdots\cdots\cdots(76)$$

in which

$$\left.\begin{aligned}
&D_\nu=-\sum_{m=1,3,5,\cdots}^{\infty}\frac{2}{m\pi}\left(\frac{\omega}{\omega_g}\right)^2\Omega_m G_\nu\\
&\Theta_{\nu\mu}=-\sum_{m=1,3,5,\cdots}^{\infty}\xi^2{}_m\Omega_m G_\nu G_\mu
\end{aligned}\right\}\quad\cdots\cdots\cdots(77)$$

and from eq. (66) we get

$p(z)=0$ $\cdots\cdots\cdots(78)$

In this case Ω_m takes the same form as in eq. (49) since the vibration of a flexible thin plate, unlike the vibration of rigid thin plate, does not restrict the motion of surface layer, the soil reaction $p(z)$ due to earth pressure tends to zero.

The natural frequencies ω_μ ($\mu=1,2,3,\cdots\cdots$) of the cylinder in the air and the fundamental natural frequency ω_g of the surface layer are related to both the embedded part and the cross section of the cylinder; *i.e.*,

$$\frac{\omega_1}{\omega_g}=C_e^*\frac{(k_1H)^2}{H/b}\sqrt{1-\frac{a_0}{a}\left(\frac{b_0}{b}\right)^3},\cdots\cdots,\quad \frac{\omega_\mu}{\omega_g}=\left(\frac{k_\mu H}{k_1 H}\right)^2\frac{\omega_1}{\omega_g},\quad\cdots\cdots \qquad\cdots\cdots\cdots(79)$$

for the case of vibration in the direction of the the minor axis, and

$$\frac{\omega_1}{\omega_g}=C_e^*\frac{(k_1H)^2}{H/a}\sqrt{1-\left(\frac{a_0}{a}\right)^3\frac{b_0}{b}},\quad\cdots\cdots,\frac{\omega_\mu}{\omega_g}=\left(\frac{k_\mu H}{k_1 H}\right)^2\frac{\omega_1}{\omega_g},\cdots\cdots \qquad\cdots\cdots\cdots(80)$$

for the case of vibration in the direction of the major axis, where C_e^* is a coefficient including the non-dimensional physical constants E_p/E, ρ/ρ_p^*, $\cdots\cdots$, etc.

(4) Characteristic Numbers a_1, b_1 and Coefficients $A_{2r+1}^{(1)}$, $B_{2r+1}^{(1)}$

The separation constants a_{2n}, a_{2n+1}, b_{2n+1} and b_{2n+2} are the so-called characteristic numbers. These characteristic numbers are the functions of the arguments q_1, q_2 and m. Only the characteristic numbers a_1, b_1 are necessary in this study. The outlines of a_1, b_1 are as follows;

When q is extremely small,

$$\begin{aligned}a_1=1+q-\frac{1}{8}q^2-\frac{1}{64}q^3-\frac{1}{1\,536}q^4+\frac{11}{36\,864}q^5+\frac{49}{589\,824}q^6\\+\frac{55}{9\,437\,184}q^7-\frac{265}{113\,246\,208}q^8+0(q^9)\end{aligned}\qquad\cdots\cdots\cdots(81)$$

$$\begin{aligned}b_1=1-q-\frac{1}{8}q^2+\frac{1}{64}q^3-\frac{1}{1\,536}q^4-\frac{11}{36\,864}q^5+\frac{49}{589\,824}q^6\\-\frac{55}{9\,437\,184}q^7-\frac{265}{113\,246\,208}q^8+0(q^9)\end{aligned}\qquad\cdots\cdots\cdots(82)$$

then

$$\left.\begin{aligned}A_1^{(1)}&=1\\A_3^{(1)}&=-\frac{1}{8}q\left\{1+\frac{1}{8}q+\frac{1}{192}q^2-\frac{11}{4\,608}q^3+\cdots\cdots\cdots\right\}\\A_5^{(1)}&=\frac{1}{192}q^2\left\{1+\frac{1}{6}q+\frac{1}{128}q^2+\cdots\cdots\cdots\right\}\\A_7^{(1)}&=-\frac{1}{9\,216}q^3\left\{1+\frac{3}{16}q+\cdots\cdots\cdots\right\}\\A_9^{(1)}&=\frac{1}{737\,280}q^4\left\{1+\cdots\cdots\cdots\right\}\end{aligned}\right\}\qquad\cdots\cdots\cdots(83)$$

and

$$\left.\begin{aligned}B_1^{(1)}&=1\\B_3^{(1)}&=-\frac{1}{8}q\left\{1-\frac{1}{8}q+\frac{1}{192}q^2-\frac{1}{4\,608}q^3+\cdots\cdots\cdots\right\}\\B_5^{(1)}&=\frac{1}{192}q^2\left\{1-\frac{1}{6}q+\frac{1}{128}q^2+\cdots\cdots\cdots\right\}\\B_7^{(1)}&=-\frac{1}{9\,216}q^3\left\{1-\frac{3}{16}q+\cdots\cdots\cdots\right\}\\B_9^{(1)}&=\frac{1}{737\,280}q^4\left\{1-\cdots\cdots\cdots\right\}\end{aligned}\right\}\qquad\cdots\cdots\cdots(84)$$

On the other hand, when q is rather large, the characteristic numbers a_1, b_1 can be obtained as

$$a_1=1+q+\frac{q^2}{a_1-9}\;\frac{q^2}{-a_1-25}\;\frac{q^2}{-a_1-49}-\cdots\cdots \qquad \cdots\cdots\cdots(85)$$

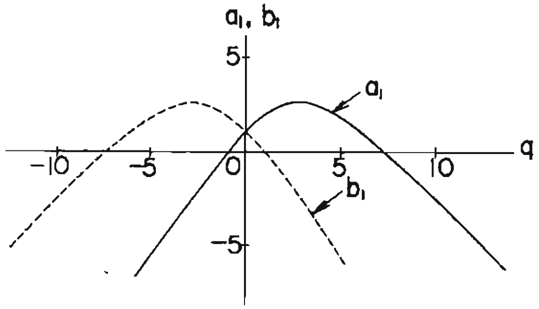

Fig. 3 Values of Characteristic Numbers a_1, b_1 for q

$$b_1=1-q+\frac{q^2}{b_1-9}\;\frac{q^2}{-b_1-25}\;\frac{q^2}{-b_1-49}-\cdots\cdots \qquad \cdots\cdots\cdots(86)$$

Relations between the characteristic numbers a_1, b_1 are conjugate for the same values of q as shown in Fig. 3. Now putting $A_3^{(1)}/A_1^{(1)}=v_1$, $A_5^{(1)}/A_3^{(1)}=v_3$, $A_7^{(1)}/A_5^{(1)}=v_5, \cdots\cdots\cdots$ gives

$$\left.\begin{aligned} &v_1=\frac{a_1-1-q}{q},\quad v_3=\frac{a_1-9}{q}-\frac{1}{v_1},\quad \cdots\cdots\cdots,\\ &v_{2r+1}=\frac{a_1-(2r+1)^2}{q}-\frac{1}{v_{2r-1}}\quad (r\geqq 1),\quad \cdots\cdots\cdots \end{aligned}\right\} \qquad \cdots\cdots\cdots(87)$$

From these relations, $A_1^{(1)}:A_3^{(1)}:A_5^{(1)}:\cdots\cdots\cdots\cdots\cdots$ can be obtained.

In the same way $B_1^{(1)}:B_3^{(1)}:B_5^{(1)}:\cdots\cdots\cdots\cdots\cdots$ can be derived with the aid of the following relations by putting $B_3^{(1)}/B_1^{(1)}=u_1$, $B_5^{(1)}/B_3^{(1)}=u_3$, $B_7^{(1)}/B_5^{(1)}=u_5, \cdots\cdots$.

$$\left.\begin{aligned} &u_1=\frac{b_1-1+q}{q},\quad u_3=\frac{b_1-9}{q}-\frac{1}{u_1},\quad \cdots\cdots\cdots,\\ &u_{2r+1}=\frac{b_1-(2r+1)^2}{q}-\frac{1}{u_{2r-1}}\quad (r\geqq 1),\quad \cdots\cdots\cdots \end{aligned}\right\} \qquad \cdots\cdots\cdots(88)$$

Now we shall normalize the Mathieu functions $ce_1(\eta, q_2)$, $se_1(\eta, q_1)$ for the case of vibration in the direction of the minor axis as follows:

$$\frac{1}{\pi}\int_0^{2\pi}\left\{ce_1(\eta, q_2)\right\}d\eta=\frac{1}{\pi}\int_0^{2\pi}\left\{se_1(\eta, q_1)\right\}^2 d\eta=1 \qquad \cdots\cdots\cdots(89)$$

Then we get

$$\sum_{r=0}^{\infty}(A_{2r+1}^{(1)})^2=\sum_{r=0}^{\infty}(B_{2r+1}^{(1)})^2=1 \qquad \cdots\cdots\cdots(90)$$

in which the coefficients $A_{2r+1}^{(1)}$ and $B_{2r+1}^{(1)}$ consist of q_2, m and q_1, m, respectively. Even for the case of vibration in the direction of the major axis, the eqs. (81) ~ (90) may be used in common with the vibration in the other direction, though the coefficients $A_{2r+1}^{(1)}$ and $B_{2r+1}^{(1)}$ consist of q_1, m and q_2, m, respectively. The Mathieu functions rapidly converge by use of these normalized coefficients $A_1^{(1)}$, $A_3^{(1)}$, $A_5^{(1)}$, $\cdots\cdots$ and $B_1^{(1)}$, $B_3^{(1)}$, $B_5^{(1)}$, $\cdots\cdots$. In the case of $q<0$, however, the Mathieu functions $ce_1(\eta, -q)$, $se_1(\eta, -q)$ can be treated in the same way as in the case of $q>0$.

3. Consideration of Numerical Computation

Since the principal aim in this study is to know the effect of the shape of a foundation structure in elastic ground, the numerical computations have been made for varying shape parameters b/a, a/H, and the ratio of wave velocity v_t/v_l, transverse to longitudinal, has been fixed at 1/3. The numerical computation of the amplification factor for the horizontal displacement at the top of the cylinder have been made only for the case where the total length of the elliptic cylinder is equal to the thickness of surface layer.

1) Consideration of the Rocking Vibration of the Elliptic Cylinder

From Fig. 1 (a), we set $H_s=2H_g=H$. The coefficients C_1^*, and C_2^* in eqs. (51)~(54) are fixed to 0.02 and 1 respectively. From eqs. (41) and (45), it is considered that the quantities f_1+if_2 and g_1+ig_2 express the dynamic effect of the horizontal spring constant and the effect of earth pressure due to the surface layer, respectively.

(1) When a/H=constant

Figs. 4, 5 and Figs. 6, 7 are respectively, the plots of f_1+if_2 and g_1+ig_2 for various values of b/a. For the vibration in the direction of the minor axis as shown in Fig. 4, the real part f_1 slowly decreases in the case of $\omega>\omega_g$ and sometimes becomes negative. On the other hand, the imaginary part f_2 sharply increases for $\omega>\omega_g$. These tendencies are more remarkable as the cross section tends to a plate which is a degenerate form of ellipse. This would mean that the response of the cylinder becomes much greater as the cross section becomes thinner because a thin rigid plate is readily subjected to the influence of the surface layer.

The imaginary part f_2 is so predominant in f_1+if_2 with increasing frequency that f_1+if_2 will behave like a damping factor. Namely, it follows that the surface layer has a damping effect upon the rigid foundation structure, because the phase difference $\tan^{-1}(f_2/f_1)$ between displacements of the bedrock and the cylinder tends to $\pi/2$ with the increase of frequency. Besides in such a case as the absolute value $|f_1+if_2|$ is mostly influenced by the imaginary part f_2, it has the effect of diminishing the amplification factor in eq. (43).

As to the vibration, however, in the direction of the major axis shown in Fig. 5, f_1+if_2 exhibits little damping effect, since the imaginary part f_2 increases so slowly with the increasing frequency of excitation in the region $\omega>\omega_g$ even for the thin cross section and the real part f_1 is a constant value independent of the frequency of excitation. In this case the phase difference $\tan^{-1}(f_2/f_1)$ between the response of the structure and the disturbing force also approaches the value of $\pi/2$ with the increase of the frequency of excitation.

Fig. 6 shows the effect upon the cylinder due to earth pressure. The real part g_1 increases sharply when the frequency ω of excitation is close to the natural frequency ω_g of the surface layer. When $\omega>\omega_g$ the value of g_1 turns from positive to negative and the absolute value $|g_1|$ rapidly decreases. Consequently in such a case, the rigid foundation structure is subjected to the earth pressure with an opposite phase to the bedrock motion. On the other hand when $\omega>\omega_g$ the imaginary part g_2 also rapidly decreases and the effect of earth pressure approaches a constant value with the increase of the frequency of excitation. Regarding the vibration in the direction of the major axis shown in Fig. 7, it appears that earth pressure has no relation with the difference in the cross section of the foundation structure but has an almost constant tendency of response.

We can obtain the frequency response of the rigid elliptic cylinder by making

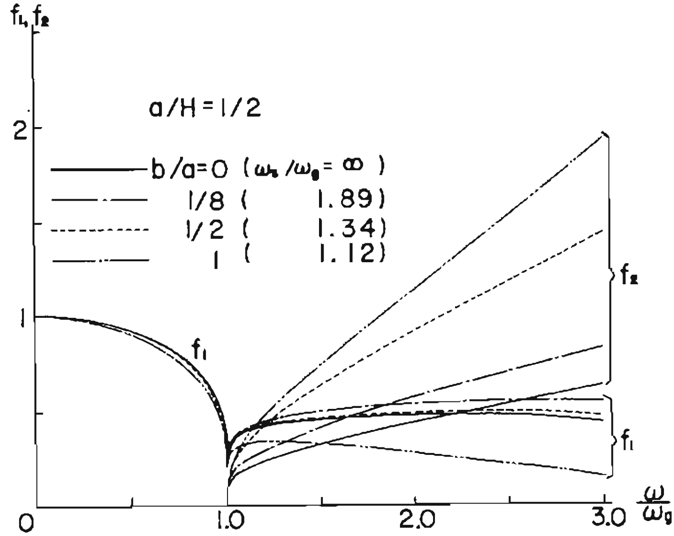

Fig. 4 Dynamic Effects of Horizontal Ground Coefficient
(in the Direction of the Minor Axis)

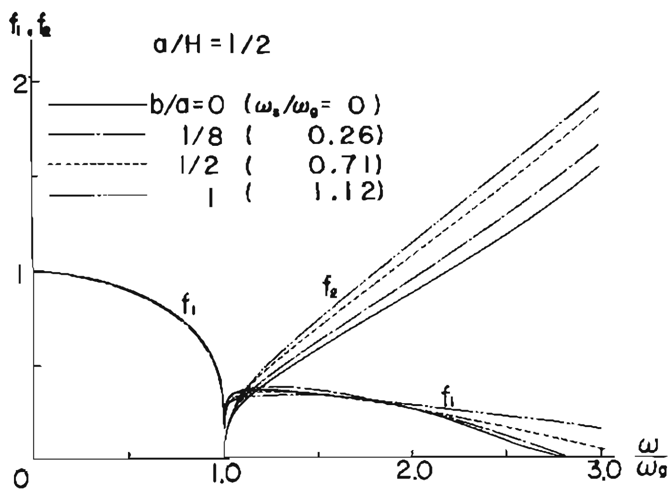

Fig. 5 Dynamic Effects of Horizontal Ground Coefficient
(in the Direction of the Major Axis)

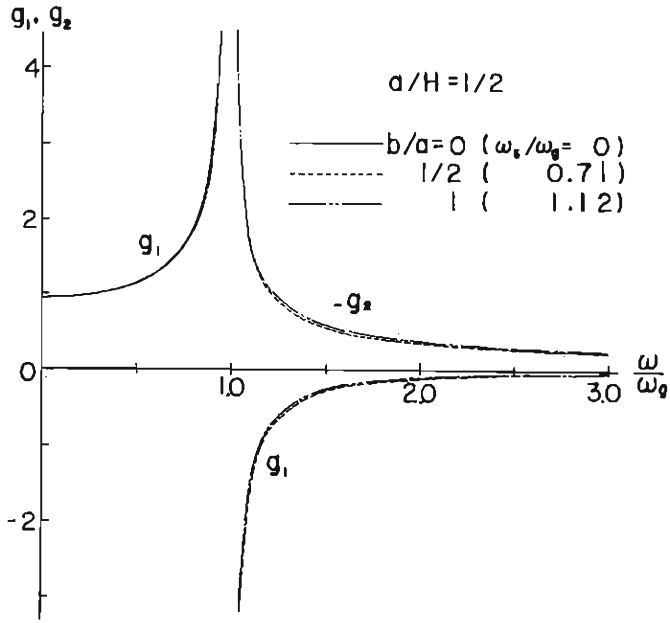

Fig. 6 Dynamic Effects of Horizontal Earth Pressure (in the Direction of the Minor Axis)

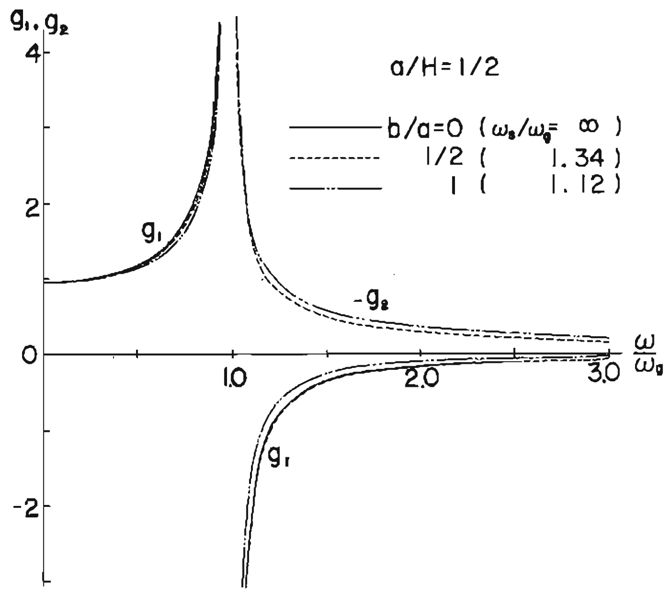

Fig. 7 Dynamic Effects of Horizontal Earth Pressure (in the Direction of the Major Axis)

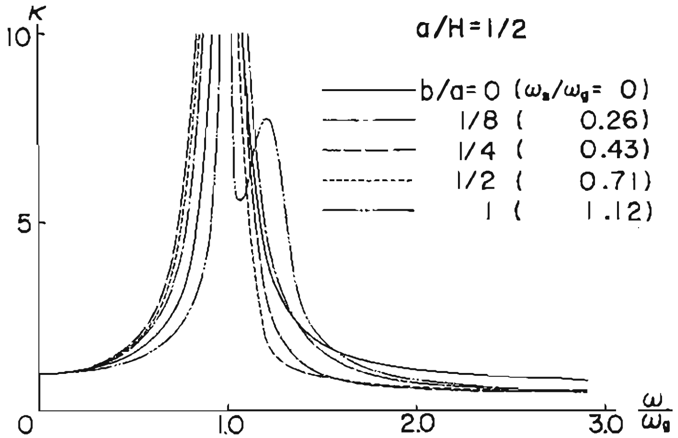

Fig. 8 Frequency Response Curves
(in the Direction of the Minor Axis)

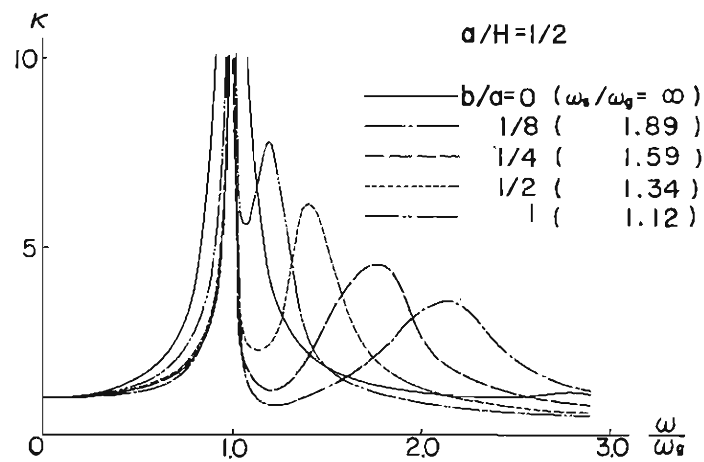

Fig. 9 Frequency Response Curves
(in the Direction of the Major Axis)

use of f_1+if_2, g_1+ig_2 as shown in Fig. 8 and Fig. 9. Regarding the vibration in the direction of the minor axis, when the natural frequency ω_s of the rigid elliptic cylinder is smaller than the fundamental natural frequency ω_g of the surface layer, the peak of the resonance curve is not clear about ω_s, but such a value of ω_s has an effect broadening the resonance peak due to the fundamental frequency of the surface layer. In such a case, the elliptic cylinder sometimes produces a greater resonance than the thin plate which is easily subjected to the influence of the surface layer. As the frequency ω of the excitation becomes larger, the response tends to be constant regardless of the cross sectional shape of the rigid elliptic cylinder.

As we see in Fig. 8, the natural frequency ω_s for the vibration in the direction of the major axis tends to infinity as the shape of the cross section gradually becomes flat. In this case, the cylinder is not greatly affected by the resonance of the surface layer but mainly by the resonance at $\omega \simeq \omega_s$. The peak of the resonance curve at $\omega \simeq \omega_s$ becomes much higher and narrower as ω_s approaches ω_g. On the other hand as ω_s diverges from ω_g, the peak of resonance at $\omega = \omega_s$ gradually decreases and the region of the resonance due to $\omega \simeq \omega_s$ becomes broader. Besides, the response of the cylinder becomes independent of the shape of the cross section as the frequency of the disturbing force increases.

(2) When b/a=constant

When the parameter a/H varies, in which H is the length of an embedded part of the foundation structure, Fig. 10 shows that the effect of rigidity of spring due to soil reaction is greatly dependent upon the thickness of the surface layer. As the frequency of excitation increases, the real part f_1 which eventually takes on negative values slowly decreases. On the other hand, the imaginary part f_2 sharply increases in proportion to the frequency of the input source. This tendency is remarkable for a stumpy cylinder. Thus f_1+if_2 has the damping effect because it becomes mainly dependent on the imaginary part f_2 as the cross section of the cylinder varies from circle to thin plate. In this case such a tendency becomes more remarkable as the thickness of the surface layer becomes smaller.

In the case of vibration in the direction of the major axis as in Fig. 11, when $\omega > \omega_g$ the real part f_1 remains nearly constant and the imaginary part f_2 increases in proportion to ω/ω_g. As in Fig. 10 the imaginary part f_2 tends to decrease as the real part f_1 becomes larger.

Anyhow from this viewpoint it follows that in the case of vibration in the direction of the minor axis, f_1+if_2 has a great damping effect upon the vibration of a stumpy cylinder, because the phase difference $\tan^{-1}(f_2/f_1)$ comes nearer to $\pi/2$. Judging from Figs. 12, 13, on the other hand, even for vibration in any direction the dynamic effect of earth pressure is not so remarkable.

The real part g_1, however, becomes much larger and the imaginary part g_2 becomes much smaller as H increases. This suggests that the dynamic effect of earth pressure increases because the phase difference $\tan^{-1}(g_2/g_1)$ becomes smaller with the increase of H. As in the case of 3. (1), the absolute values of both the real part g_1 and the imaginary part g_2 approach constant values in the region of $\omega > \omega_g$ as ω increases.

Making use of these values, the diagrams of the amplification factor at the top of the cylinder are shown in Figs. 14, 15 for the various values of parameter a/H. In the case of vibration in the direction of the minor axis there is no

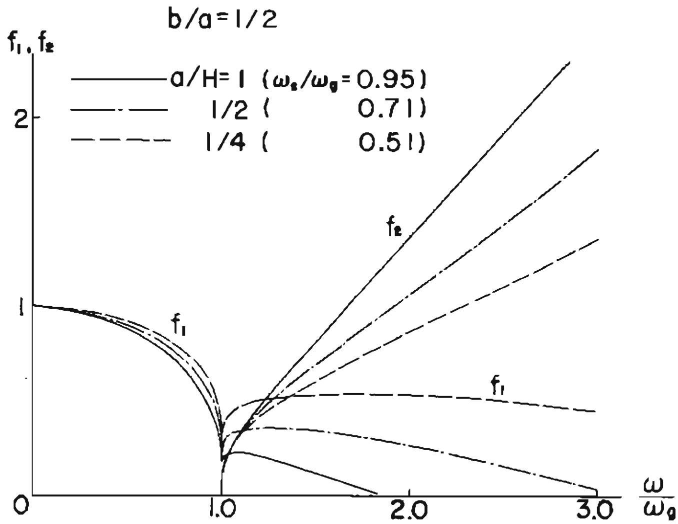

Fig. 10 Dynamic Effects of Horizontal Ground Coefficient (in the Direction of the Minor Axis)

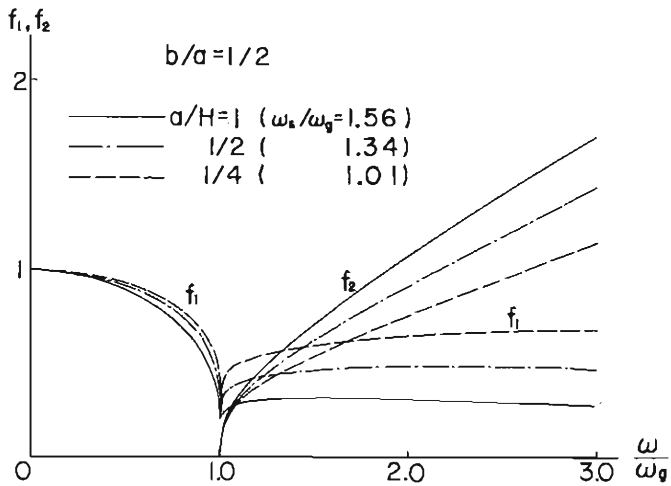

Fig. 11 Dynamic Effects of Horizontal Ground Coefficient (in the Direction of the Major Axis)

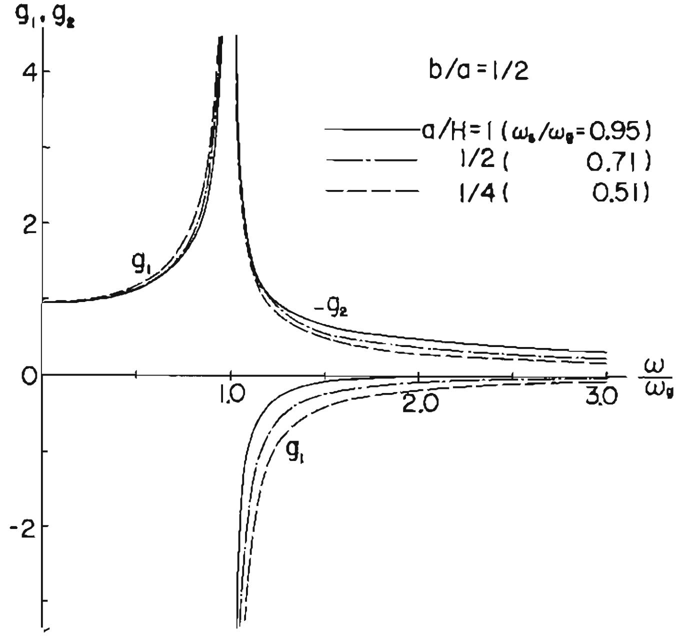

Fig. 12 Dynamic Effects of Horizontal Earth Pressure (in the Direction of the Minor Axis)

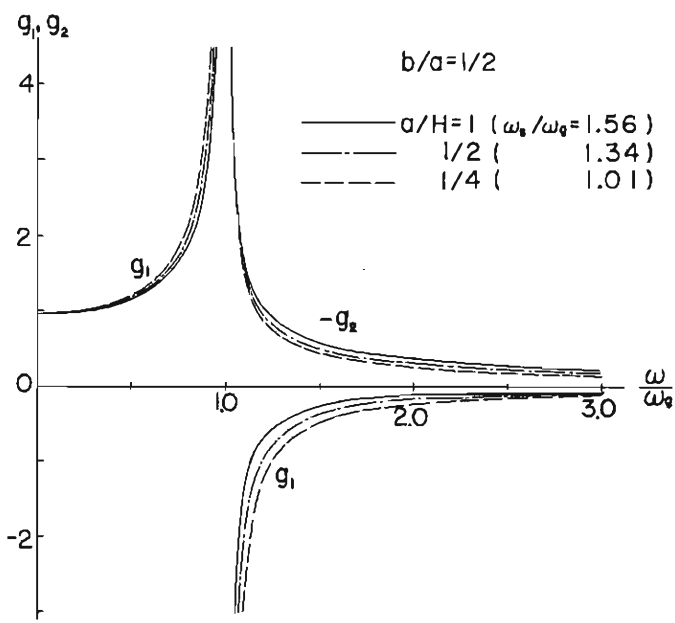

Fig. 13 Dynamic Effects of Horizontal Earth Pressure (in the Direction of the Major Axis)

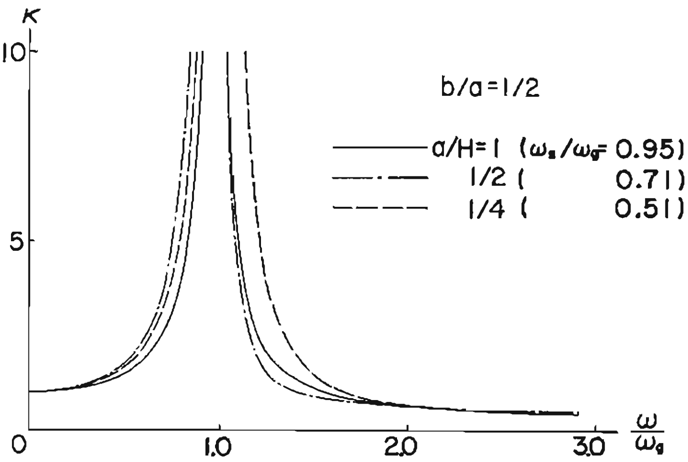

Fig. 14 Frequency Response Curves
(in the Direction of the Minor Axis)

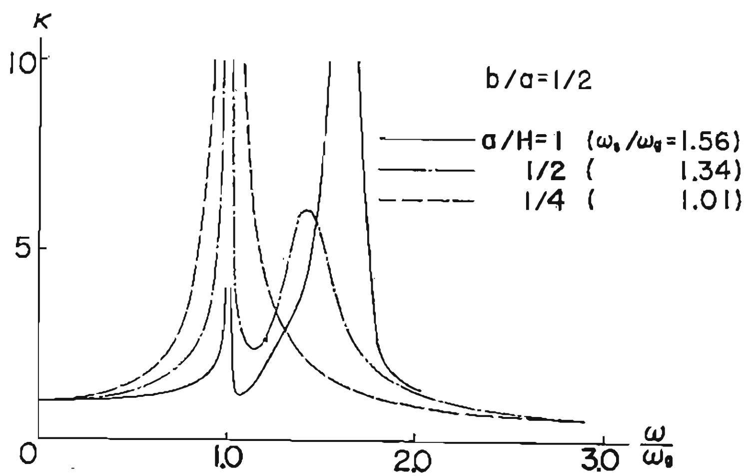

Fig. 15 Frequency Response Curves
(in the Direction of the Major Axis)

resonance due to $\omega=\omega_s$ but the effect of broadening the resonance region of the surface layer due to $\omega=\omega_g$. Such a condition is similar to the response of the thin plate which is easily subjected to the influence of the surface layer. It is, however, considerably different from the response for the case where a/H is constant, that the response of the cylinder rapidly decreases and comes in a constant value when $\omega<\omega_g$, no matter what resonance due to $\omega=\omega_g$ is very large.

On the other hand in the case of vibration in the direction of the major axis the value of ω_s decreases and resonance due to $\omega=\omega_g$ increases as H increases. Diminution of H, however, has not so much effect upon the resonance of the surface layer but a great effect upon the resonance due to $\omega=\omega_s$. This means that the effect of soil reaction due to the surface layer decreases as a/H becomes smaller, though the effect due to the reaction of the bedrock increases. Thus the bedrock will sometimes have a great effect upon the response of the foundation structure for the case of the shallow embedded part of the structure. Consequently it follows that the great or small depth of the embedded part has a considerably different influence upon the response of the foundation structure even for the case of the foundation structure with the same cross section.

2) Consideration of the Flexural Vibration of the Elliptic Cylinder

Figs. 16, 17, 18 and 19 are the diagrams of the frequency response curves for the different values of the parameter E_p/E_g, which is the ratio of the elastic constants of the foundation structure to that of the surface layer. The coefficient C_e^* shows the degree of hardness or softness of the surface layer in comparison with the cylinder. In these numerical computations the inner part of the cylinder is treated as being filled with the same soil as that surrounding the cylinder. The thickness of the cylinder is always kept at 2/3 of the outside half length a of the maior axis. For facility in making the numerical computation, the first and second mode are taken into account. These figures show that the fundamental natural frequency ω_1 of the cylinder itself is generally smaller than ω_1^* of this system, which consists of the cylinder and surface layer, except for the extremely large value of E_p/E_g. This tendency is very different from the case where ω_1 is always greater than ω_1^* for the case of the submerged cylinder[5]. Such a different phenomenon may be explained as follows. When the cylinder is submerged, the water acts on the cylinder only as an inertia force, so that the water behaves as if it were the additional mass to the cylinder. Consequently the fundamental natural frequency ω_1^* of the water-cylinder system becomes smaller than ω_1 of the cylinder in the air. On the other hand, Figs. 16~19 indicate that the ground surrounding the cylinder would behave rather as a spring than additional mass when the cylinder is embedded in the elastic ground. Accordingly ω_1^* tends to be larger than ω_1. This fact will possiblly ensure that in the analysis of the foundation structure embedded in the ground, the ground around the foundation structure may be estimated as the mechanical spring without the mass.

Figs. 16, 17 may correspond to the diagrams for relatively hard ground. Either in the case for the vibration in the direction of the minor axis or the major axis, as shown in Figs. 16, 17, ω_1^* is about 130% larger than ω_1 for the case of the circular cylinder. The surface layer has a greater effect upon the cylinder than in the case for the rocking vibration, though the resonance due to $\omega=\omega_1^*$ is not so remarkable as for the rocking vibration. This means that the flexible cylinder is more easily affected by the vibrational mode of the surface layer

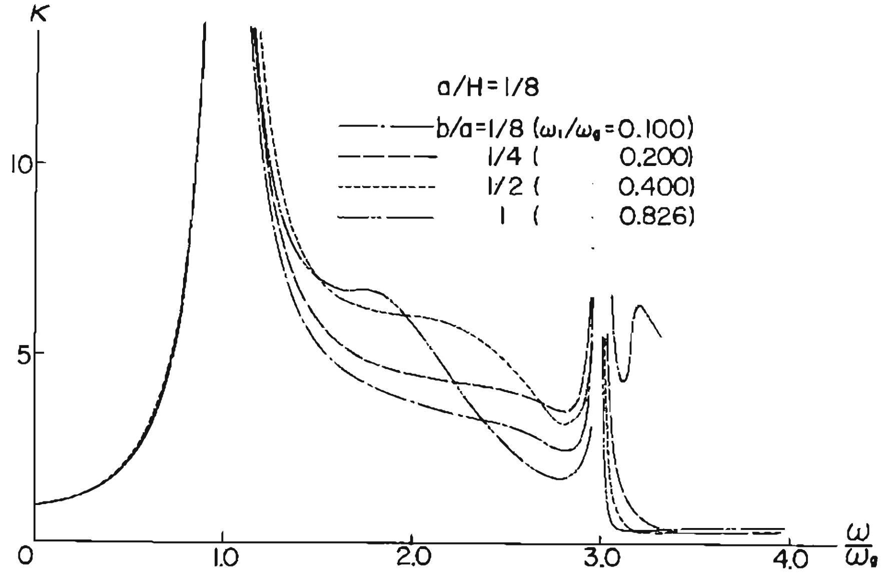

Fig. 16 Frequency Response Curves
(in the Direction of the Minor Axis; $Ce^{*}=3.65$)

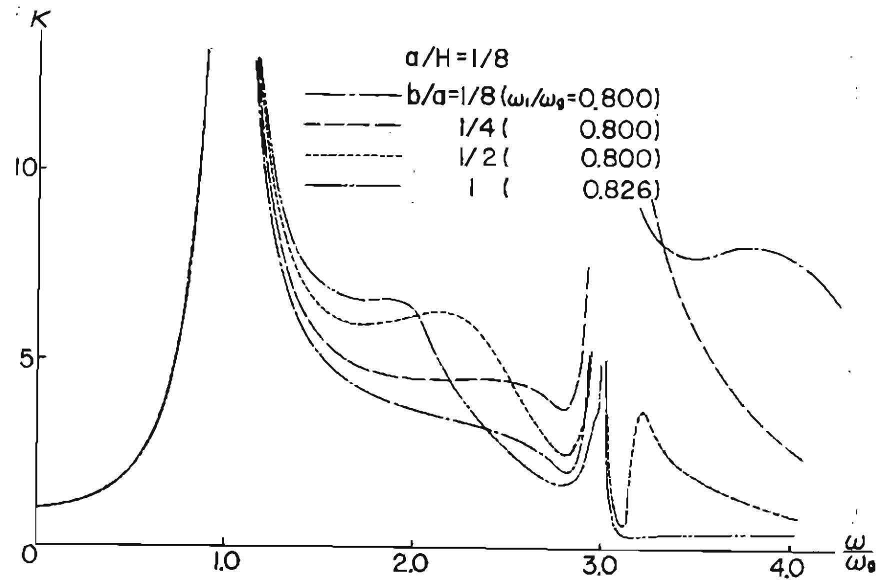

Fig. 17 Frequency Response Curves
(in the Direction of the Major Axis; $Ce^{*}=3.65$)

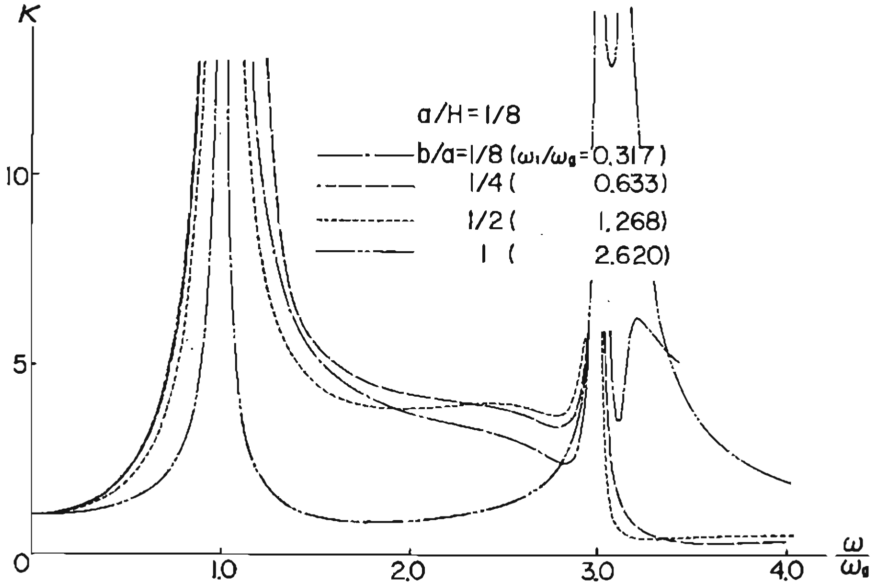

Fig. 18 Frequency Response Curves
(in the Direction of the Minor Axis; $Ce^{*}=11.55$)

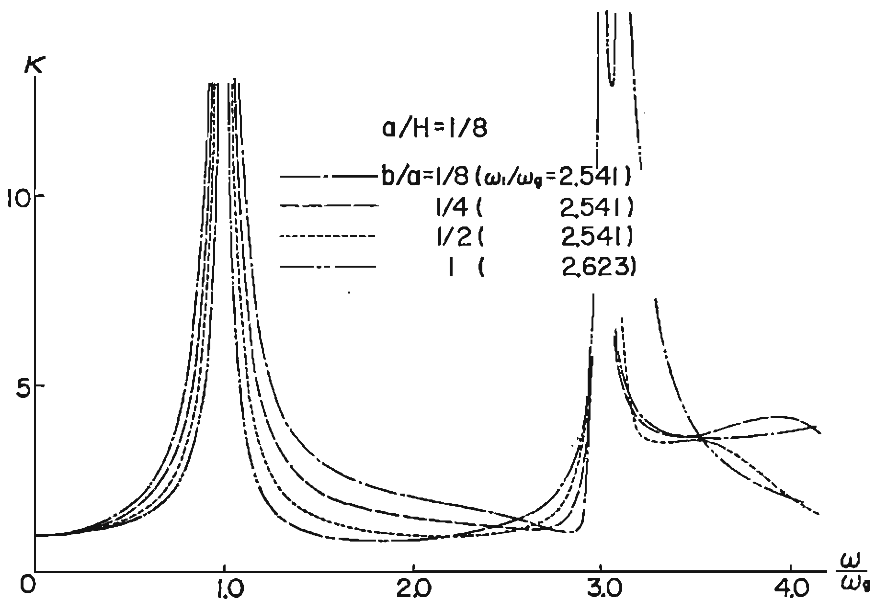

Fig. 19 Frequency Response Curves
(in the Direction of the Major Axis; $Ce^{*}=11.55$)

than the rigid cylinder. In the case of the vibration in the direction of the minor axis as in Fig. 16, $\omega_1{}^*$ also rapidly decreases because the rigidity of the cylinder quickly decreases with the flatness of the cross section. When the cylinder becomes so flexible as to be $\omega_1{}^* < \omega_g$, the resonance of the cylinder owing to $\omega \simeq \omega_1{}^*$ does not occur because of being subjected to the dominant influence of the surface layer.

On the other hand in the case of the vibration in the direction of the major axis, as illustrated in Fig. 17 the variation of the cross section has not so much effect upon the rigidity of the cylinder. In spite of such a tendency, the increase of $\omega_1{}^*$ accompanied by the flatness of the cross section would arise owing to the fact that transverse vibration is predominant rather than longitudinal vibration in the behaviour of the surface layer, namely that the surface layer acts just like the spring rather than the additional mass.

Figs. 18, 19 are the diagrams of resonance curves in which the ratio E_p/E_g is 10 times as much as that in Figs. 16 and 17. In this case $\omega_1{}^*$ is about 20% larger than ω_1 for the case of the circular cylinder. We know that the fundamental natural frequency $\omega_1{}^*$ of this system approaches the fundamental natural frequency ω_1 of the cylinder in the air, when the surface layer becomes considerably softer. In the case of the vibration in the direction of the minor axis, the resonance owing to $\omega \simeq \omega_1{}^*$ is not so distinct but the response of the cylinder becomes noticeably greater with the flatness of the cross section when $\omega_1 > \omega_g$. This clearly suggests that the soil reaction tends to be easily subjected to the influence of the shearing vibration with the flatness of the cross section for the vibration in the direction of the minor axis.

On the other hand the fundamental natural frequency $\omega_1{}^*$ of this system exceeds $3\omega_g$ in the case of the vibration in the direction of the major axis as in Fig. 18. In such a case where $\omega/\omega_g < 3$, the response of the cylinder takes a small value as the cross section approaches a circle. The difference in the cross section, however, has not so much effect upon the response of the cylinder as in the case of the vibration in the direction of the minor axis.

4. Concluding Remarks

In the analysis of the present study, we have expanded the response analysis for a foundation structure with a circular cross section developed by Prof. Tajimi to a general form governing the dynamic behaviour of foundation structures with various cross sectional shapes. Besides, it has been shown from the numerical computations that not only difference in the embedded length but the difference in the cross section leads to different responses of foundation structures.

When the foundation structure is rigid, the following conclusions are obtained:-

(1) Different direction of vibration gives a different response of the foundation structure except for a circular cylinder.

(2) In the case of vibration in the direction of the minor axis, the damping effect of soil reaction increases not only as the shape of the cross section becomes flatter but as the embedded length becomes smaller.

(3) As the embedded length of a structure becomes greater, the effect of earth pressure increases, though the difference in shape of the cross section has not so

much influence upon the earth pressure.
(4) When the natural frequency of the foundation structure is smaller than that of the surface layer, resonance of foundation structure does not occur even when the frequency of excitation coincides with the natural frequency of the structure; otherwise there is resonance.
(5) In the case of vibration in the direction of the major axis, the damping effect which the surface layer exerts on the vibration of the structure is remarkable for a stumpy structure.

On the other hand when the foundation structure is flexible, the following conclusions are obtained:-
(1) The natural frequency of the system is generally larger than that of the structure in the air. When the surface layer becomes relatively soft, both natural frequencies approach each other.
(2) The great resonance of the foundation structure will take place when the exciting frequency coincides not with the fundamental natural frequency of structures but with that of the surface layer.
(3) As in the case of the rocking motion of a rigid structure, the flexible structure is also greatly affected by the surface layer, with flattening of the cross section in the case of vibration in the direction of the minor axis.
(4) For the case where the fundamental natural frequency of the structure becomes larger than the second natural frequency of the surface layer, the response of the structure tends to be remarkably small.

In these theoretical analyses and numerical computations, the superstructure of the foundation structure is not taken into account. It is clear, however, that such a problem can be treated in the same way as our analysis. When the seismic response of foundation structure with the elliptic cross section is analyzed for excitation in an arbitrary direction, we can obtain the exact method of aseismic design for foundation structures in the ground.

Finally, it is acknowledged that the numerical computations were carried out on digital computer KDC-II of the Kyoto University Computation Center.

Notations

$u_p(z, t)$	: relative displacement of elliptic cylinder to the bedrock at an arbitrary height z from the bedrock
a, b	: half lengths, respectively, of the major and minor axis of ellipse
k	: $\sqrt{a^2-b^2}$ $(a>b)$
ξ_0	: coordinate on the surface of elliptic cylinder
u_0	: maximum displacement of bedrock
λ, μ	: Lamé's constants
ρ	: density of surface layer
ω	: frequency of excitation
ω_g	: fundamental natural frequency of surface layer
ω_s	: fundamental natural frequency of rigid elliptic cylinder
Δ	: dilatation
$\tilde{\omega}_\xi$, $\tilde{\omega}_\eta$, $\tilde{\omega}_z$	: rotations around ξ, η, z axes, respectively
θ	: angle between x- axis and a tangent to the hyperbola

u_ξ, u_η, u_z : displacements in the direction of ξ, η, z, respectively
σ_ξ, σ_η : normal stresses in the direction of ξ, η, respectively
$\tau_{\xi\eta}$: shearing stress in the $\xi-\eta$ plane
Δ^2 : Laplacian
Φ, Ψ : potential functions concerning displacements
H : thickness of surface layer
v_l, v_t : velocity, respectively, of longitudinal and transverse waves in surface layer
λ : separation constant
$R(\xi)$, $\Theta(\eta)$: functions of single variable, ξ, η, respectively
a_{2n+1}, a_{2n}, b_{2n+1}, b_{2n+2} : characteristic numbers concerning the Mathieu functions
$A_{2r+1}^{(1)}$, $B_{2r+1}^{(1)}$: coefficients peculiar to the Mathieu functions
C_m, D_m, C'_m, D'_m : integration constants
s : maximum odd positive integer satisfying $\xi_m{}^2>0$
φ : angular amplitude of rigid elliptic cylinder
φ_0 : maximum angular amplitude of rigid elliptic cylinder
$u_{p,\xi}$, $u_{p,\eta}$: displacements of rigid elliptic cylinder in the directions of ξ, η, respectively
$p(z)$: soil reaction due to earth pressure per unit length of elliptic cylinder
M : overturning moment of rigid elliptic cylinder around the center line of the bottom
m_0 : total mass of rigid elliptic cylinder
H_s : total height of rigid elliptic cylinder
I : moment of inertia around the center line of the bottom of the cylinder
k_r : spring constant for rocking vibration
χ : ratio of static horizontal spring constant to that for rocking vibration
$\kappa(z)$: amplification factor for rigid elliptic cylinder at an arbitrary height from bedrock
$\rho_b{}^*$: density of bedrock
$v_t{}^*$: velocity of transverse wave in bedrock
$I_0{}^*$: geometrical moment of inertia at the bottom surface of rigid body
A : bottom area of rigid body
ν_b : Poisson ratio of bedrock
$C_1{}^*$, $C_2{}^*$: coefficients containing various physical constants
$\rho_p{}^*$: equivalent density of elliptic cylinder estimated to the total cross section
E : Young's modulus of flexible elliptic cylinder for real cross section neglecting inner section
I_0 : geometrical moment of inertia for real cross section neglecting inner section
a_0, b_0 : half lengths, respectively, of the major axis and minor axis in inner cross section of elliptic cylinder
$U_p(z)$: deflection of foundation structure

A_μ	: undetermined coefficient concerning characteristic function
$\eta_\mu(k_\mu z)$	: characteristic function of cantilever
$k_\mu H$	: characteristic value of cantilever
ω_μ	: natural frequency of cantilever in μ- th order
$\omega_\mu{}^*$	: natural frequency of the surface layer-flexible cylinder system

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